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METHODS OF MATRIX INVERSION

DONALD GREENSPAN, University of Maryland

Introduction. The physical scientist has long been plagued with large linear systems of equations, hopeless differential equations or systems of such equations, or simply enigmatic integral equations. Each of these problems in turn often lends itself to matrix analysis and the approach usually involves matrix inversion. However, it is only with the advent of high speed computation that the approach has entered the realm of practicality.

The purpose of this paper is to present some of the existing methods of matrix inversion in a fashion requiring a minimal mathematical background. ("Minimal mathematical background" implies knowing the definition of a matrix, how to add and multiply matrices, how to apply matrix notation to systems of linear equations, and how to evaluate a determinant. Most of this material is discussed in [27].)

In each of the following discussions, matrix A will be assumed non-singular and will be represented by:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots a_{3n} \\ \vdots & & \\ a_{n1} & a_{n2} & a_{n3} \cdots a_{nn} \end{bmatrix}$$

The main diagonal of A will be that part of the matrix consisting of the elements a_{ii}, $i=1, 2, \cdots, n$; and the trace $[A]$ will be the sum of these elements. The unit matrix is one such that $a_{ii}=1$; $i=1, 2, \cdots, n$; and $a_{ij}=0$; $i \neq j$; $i, j=1, 2, 3, \cdots, n$. A is called symmetric if and only if $a_{ij}=a_{ji}$ for all i and j.

Method I—Elimination method. Multiply row 1 by $1/a_{11}$. Then multiply row 1 by $-a_{i1}$ and add to row i for $i=2, 3, 4, \cdots, n$. This sequence of operations has the effect of creating a 1 in the first position of the main diagonal and zeros everywhere else in the first column of the matrix. Apply the same technique to the second row, thus creating a 1 in the second position on the main diagonal and zeros everywhere else in the second column. Apply the same technique to the third row, again resulting in a column of $(n-1)$ zeros and a 1 in the main diagonal position. Keep up this procedure until the given matrix is reduced to the unit matrix. By performing the very same sequence of operations on the unit

matrix as were performed on A, using the very same numbers necessary when operating on A, the unit matrix is transformed into A^{-1}.

Example 1: Let

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

Then we write down:

$$\left[\begin{array}{ccc|ccc} 2 & -2 & 4 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 \end{array}\right].$$

We multiply row 1 by $\frac{1}{2}$ and secure:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & \frac{1}{2} & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 \end{array}\right].$$

By multiplying row 1 by -2 and adding to row 2 and multiplying row 1 by 1 and adding to row 3 we have:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 5 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 1 \end{array}\right].$$

Multiply row 2 by $\frac{1}{5}$. Hence:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{5} & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 1 \end{array}\right].$$

The sequence of operations: multiply row 2 by 1 and add to row 1; multiply row 2 by 0 and add to row 3; multiply row 3 by 1; multiply row 3 by $-\frac{2}{5}$ and add to row 1; multiply row 3 by $\frac{2}{5}$ and add to row 2, yields:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{5} & -\frac{8}{5} \\ 0 & 1 & 0 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 1 \end{array}\right].$$

Hence the inverse of

$$\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix} \text{ is } \begin{bmatrix} -\frac{1}{2} & \frac{1}{5} & -\frac{8}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{2} & 0 & 1 \end{bmatrix}.$$

The only problem which can arise, assuming no loss of significance to make the matrix at any stage singular, is that one of the a_{ij} becomes 0. In this case, one need only interchange two rows so that the zero on the main diagonal is removed from its choice position. In order to get A^{-1} then, one need only take the final result and interchange in it the same numbered columns as those of the rows which were interchanged.

A technique like this is called an elimination technique. Application of it to a large scale matrix should be accompanied by an examination of [30].

Method II. Consider the matrix B such that:

$$B = \left[\begin{array}{c|c} I & A \\ \hline 0 & I \end{array}\right],$$

where A is the matrix to be inverted, I is the identity matrix, and 0 the matrix composed of all zero elements. Consider the following operations:

a) multiply any row or any column by a non-zero constant.
b) multiply any row (column) by a non-zero constant and add to another row (column).
c) interchange any two rows or any two columns.

Using these operations, we reduce

$$B \to \left[\begin{array}{c|c} P & I \\ \hline 0 & Q \end{array}\right].$$

Then $A^{-1} = QP$.

Example 2:

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

Then

$$B = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & 4 \\ 0 & 1 & 0 & 2 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 2 & 3 & 2 \\ 1 & 0 & 0 & 2 & -2 & 4 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 5 & 0 \\ 1 & 0 & 0 & 2 & 0 & 2 \\ \hline 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 5 & 0 \\ 1 & 0 & 2 & 0 & 0 & 2 \\ \hline 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & \frac{1}{5} & \frac{2}{5} & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Hence

$$P = \begin{bmatrix} 0 & 0 & -1 \\ 0 & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{2} & 0 & 1 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$A^{-1} = QP = \begin{bmatrix} -\frac{1}{2} & \frac{1}{5} & \frac{8}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{2} & 0 & 1 \end{bmatrix}.$$

Let us note first that method I is a special case of method II but that method I is worthy of special discussion, for although Andree [2] claims method II is the faster of the two, method I is more readily coded for high speed machines.

Method III. Given matrix A, consider matrix B such that $AB = I$. Setting the product AB equal to I term by term yields n^2 equations in n^2 unknowns which one tries to solve. These actually form n distinct sets of linear systems.

Example 3: Given

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}, \qquad \text{let } B = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}.$$

Then:

$$\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} = \begin{bmatrix} 2x_1-2x_2+4x_3 & 2y_1-2y_2+4y_3 & 2z_1-2z_2+4z_3 \\ 2x_1+3x_2+2x_3 & 2y_1+3y_2+2y_3 & 2z_1+3z_2+2z_3 \\ -x_1+x_2-x_3 & -y_1+y_2-y_3 & -z_1+z_2-z_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence:

$$\begin{array}{lll} 2x_1 - 2x_2 + 4x_3 = 1 & 2y_1 - 2y_2 + 4y_3 = 0 & 2z_1 - 2z_2 + 4z_3 = 0 \\ 2x_1 + 3x_2 + 2x_3 = 0 & 2y_1 + 3y_2 + 2y_3 = 1 & 2z_1 + 3z_2 + 2z_3 = 0 \\ -x_1 + x_2 - x_3 = 0 & -y_1 + y_2 - y_3 = 0 & -z_1 + z_2 - z_3 = 1, \end{array}$$

the solutions of which are

$$\begin{array}{lll} x_1 = -\frac{1}{2}, & x_2 = 0, & x_3 = \frac{1}{2} \\ y_1 = \frac{1}{5}, & y_2 = \frac{1}{5}, & y_3 = 0 \\ z_1 = -\frac{3}{5}, & z_2 = \frac{2}{5}, & z_3 = 1. \end{array}$$

Therefore:

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{5} & -\frac{3}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{2} & 0 & 1 \end{bmatrix}.$$

It will be noted that the problem of matrix inversion has here been converted to one of solving linear systems of equations. Much research has been done on this latter problem and hence any technique associated with solving linear systems can be applied to matrix inversion. Besides the usual methods of: a) Cramer's rule, b) Gauss elimination method, c) Gauss-Jordan method, and d) Crout's method, more modern techniques are described in: [5], [8], [12], [16], [21], [23], [26].

Method IV. Adjoint method.

Let A_{11} = determinant formed from matrix A by deleting row 1 and column 1 and multiplying by $(-1)^{1+1}$.

Let A_{21} = determinant formed from matrix A by deleting row 2 and column 1 and multiplying by $(-1)^{2+1}$.

Let A_{37} = determinant formed from matrix A by deleting row 3 and column 7 and multiplying by $(-1)^{3+7}$.

Let A_{ij} = determinant formed from matrix A by deleting row i and column j and multiplying by $(-1)^{i+j}$.

Let C be the matrix:

$$C = \begin{bmatrix} A_{11} & A_{21} & A_{31} \cdots A_{n1} \\ A_{12} & A_{22} & A_{32} \cdots A_{n2} \\ \vdots & & \\ A_{1n} & A_{2n} & \cdots A_{nn} \end{bmatrix}.$$

Then $A^{-1} = (1/|A|)C$, where $|A|$ is the determinant of matrix A.

Example 4. Let:

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

Then:

$$A_{11} = (-1)^2 \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} = -5 \qquad A_{21} = (-1)^3 \begin{vmatrix} -2 & 4 \\ 1 & -1 \end{vmatrix} = 2$$

$$A_{12} = (-1)^3 \begin{vmatrix} 2 & 2 \\ -1 & -1 \end{vmatrix} = 0 \qquad A_{22} = (-1)^4 \begin{vmatrix} 2 & 4 \\ -1 & -1 \end{vmatrix} = 2$$

$$A_{13} = (-1)^4 \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = 5 \qquad A_{23} = (-1)^5 \begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix} = 0$$

$$A_{31} = (-1)^4 \begin{vmatrix} -2 & 4 \\ 3 & 2 \end{vmatrix} = -16$$

$$A_{32} = (-1)^5 \begin{vmatrix} 2 & 4 \\ 2 & 2 \end{vmatrix} = 4$$

$$A_{33} = (-1)^6 \begin{vmatrix} 2 & -2 \\ 2 & 3 \end{vmatrix} = 10.$$

Also:

$$|A| = \begin{vmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{vmatrix} = 10;$$

therefore

$$A^{-1} = \frac{1}{10}\begin{bmatrix} -5 & 2 & -16 \\ 0 & 2 & 4 \\ 5 & 0 & 10 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{5} & -\frac{8}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{2} & 0 & \frac{1}{10} \end{bmatrix}.$$

Method V—Method of Partitions. Given matrix A, let the sequence of matrices $S_1, S_2, \cdots, S_N$, be defined by:

$$S_1 = [a_{11}],\ S_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \left[\begin{array}{c|c} S_1 & a_{12} \\ \hline a_{21} & a_{22} \end{array}\right],\ S_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \left[\begin{array}{cc|c} & S_2 & a_{13} \\ & & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right], \cdots$$

$$S_N = A = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1N} \\ a_{21} & a_{22} \cdots a_{2N} \\ \vdots & \\ a_{N1} & a_{N2} \cdots a_{NN} \end{bmatrix} = \left[\begin{array}{c|c} S_{N-1} & \begin{matrix} a_{1N} \\ a_{2N} \\ \vdots \end{matrix} \\ \hline a_{N1}a_{N2}\cdots & a_{NN} \end{array}\right].$$

In general then, we shall want to partition a matrix into 4 sub-matrices which we write as: (See [11], pp. 112–115)

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}.$$

For example, for S_3, we have:

$$\alpha_{11} = S_2, \qquad \alpha_{12} = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}, \qquad \alpha_{21} = [a_{31}\ a_{32}], \qquad \alpha_{22} = [a_{33}].$$

If we denote the inverse of A by B and let:

$$B = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix},$$

then we have:

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \cdot \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} = I.$$

Multiplying out and setting the resulting matrix equal to unity yields four matrix equations. Solving these matrix equations for $\beta_{11}, \beta_{12}, \beta_{21}$, and β_{22}, we have:

$$\begin{aligned} \beta_{22} &= D \\ \beta_{21} &= -D(\alpha_{21}\alpha_{11}^{-1}) \\ \beta_{12} &= -(\alpha_{11}^{-1}\alpha_{12})D \\ \beta_{11} &= \alpha_{11}^{-1} + (\alpha_{11}^{-1}\alpha_{12})D\alpha_{21}\alpha_{11}^{-1}, \end{aligned}$$

where $D=(\alpha_{22}-\alpha_{21}\alpha_{11}^{-1}\alpha_{12})^{-1}$.

The technique of inversion will be to apply the above formulas in a recursive fashion by first finding the inverse of S_1, then of S_2, then of S_3, $\cdots$, finally of S_N and this will be A^{-1}. One interchanges two rows to avoid trouble when a submatrix is singular and the process cannot be continued directly. This necessitates only changing the same two columns in the result to find A^{-1}.

Example 5:

$$A=\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

Then $S_1=[2]$, $S_1^{-1}=[\frac{1}{2}]$. Hence:

$$S_2=\begin{bmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{bmatrix}=\begin{bmatrix}S_1 & a_{12}\\ a_{21} & a_{22}\end{bmatrix}=\begin{bmatrix}2 & -2\\ 2 & 3\end{bmatrix}.$$

Using the formulas for β_{ij}, it follows that: $\beta_{11}=\frac{3}{10}$, $\beta_{12}=\frac{1}{5}$, $\beta_{21}=-\frac{1}{5}$, $\beta_{22}=\frac{1}{5}$. Hence

$$S_2^{-1}=\begin{bmatrix}\beta_{11} & \beta_{12}\\ \beta_{21} & \beta_{22}\end{bmatrix}=\begin{bmatrix}\frac{3}{10} & \frac{1}{5}\\ -\frac{1}{5} & \frac{1}{5}\end{bmatrix}.$$

Now:

$$S_3=\left[\begin{array}{c|c} S_2 & \begin{matrix}a_{13}\\ a_{23}\end{matrix} \\ \hline \begin{matrix}a_{31} & a_{32}\end{matrix} & a_{33}\end{array}\right]=\begin{bmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{bmatrix}=\left[\begin{array}{cc|c} 2 & -2 & 4\\ 2 & 3 & 2\\ \hline -1 & 1 & -1\end{array}\right].$$

Then:

$$\alpha_{11}=S_2=\begin{bmatrix}2 & -2\\ 2 & 3\end{bmatrix},\qquad \alpha_{12}=\begin{bmatrix}a_{13}\\ a_{23}\end{bmatrix}=\begin{bmatrix}4\\ 2\end{bmatrix},$$

$$\alpha_{21}=[a_{31}\ a_{32}]=[-1,\ 1],\qquad \alpha_{22}=[a_{33}]=[-1].$$

Note that $S_2^{-1}=\alpha_{11}^{-1}$ is known from the previous calculation. Again using the formulas for the β_{ij}, it follows that:

$$\beta_{11}=\begin{bmatrix}-\frac{1}{2} & \frac{1}{5}\\ 0 & \frac{1}{5}\end{bmatrix},\qquad \beta_{12}=\begin{bmatrix}-\frac{8}{5}\\ \frac{2}{5}\end{bmatrix},\qquad \beta_{21}=[\tfrac{1}{2}\ 0],\qquad \beta_{22}=[1].$$

Therefore:

$$A^{-1}=S_3^{-1}=\begin{bmatrix}-\frac{1}{2} & \frac{1}{5} & -\frac{8}{5}\\ 0 & \frac{1}{5} & \frac{2}{5}\\ \frac{1}{2} & 0 & 1\end{bmatrix}.$$

Concerning experiments with this method, one should consult [17].

Method VI. Given matrix A, consider the matrix: $B=A-\lambda I$, where I is the unit matrix and λ is a constant. If we set the determinant of B equal to zero, *i.e.*, $|B|=0$, we see that this is nothing more than a polynomial equation in λ of degree n. The values $\lambda_1, \lambda_2, \cdots, \lambda_n$ which satisfy this equation are called the eigenvalues of the matrix A and the equation itself is called the characteristic equation of the matrix A. For example, if:

$$A=\begin{bmatrix}2 & -2 & 4\\ 2 & 3 & 2\\ -1 & 1 & -1\end{bmatrix}, \quad B=\begin{bmatrix}2 & -2 & 4\\ 2 & 3 & 2\\ -1 & 1 & -1\end{bmatrix}-\lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}2-\lambda & -2 & 4\\ 2 & 3-\lambda & 2\\ -1 & 1 & -1-\lambda\end{bmatrix},$$

the characteristic equation is:

$$\lambda^3-4\lambda^2+7\lambda-10=0.$$

A fundamental theorem of matrix theory says that every matrix satisfies its characteristic equation. By the example above, we explain this to mean:

$$\begin{bmatrix}2 & -2 & 4\\ 2 & 3 & 2\\ -1 & 1 & -1\end{bmatrix}^3-4\begin{bmatrix}2 & -2 & 4\\ 2 & 3 & 2\\ -1 & 1 & -1\end{bmatrix}^2+7\begin{bmatrix}2 & -2 & 4\\ 2 & 3 & 2\\ -1 & 1 & -1\end{bmatrix}$$

$$-10\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

and this may be readily verified by performing the indicated operations.

A matrix may satisfy many different equations besides its characteristic equation. It can be shown that there exists an equation of least degree which the matrix satisfies and this degree may be less than that of the characteristic equation. This equation of least degree is unique to within a multiplicative constant and is appropriately called the minimal equation associated with the matrix. If $C(x)=0$ is the characteristic equation associated with a matrix and $M(x)=0$ is the minimal equation which the matrix satisfies, then it can be shown that $M(x)$ is a factor of $C(x)$.

Now, suppose some equation has been found which the matrix satisfies, the characteristic equation usually sufficing. Then if the equation is:

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n=0, \qquad a_n\neq 0,$$

and if A is the matrix:

$$a_0A^n + a_1A^{n-1} + \cdots + a_{n-1}A + a_nI = 0.$$

Multiplying through by A^{-1}, we have:

$$a_0A^{n-1} + a_1A^{n-2} + \cdots + a_{n-1}I + a_nA^{-1} = 0,$$

or:

$$A^{-1} = \frac{1}{a_n}\left\{-a_0A^{n-1} - a_1A^{n-2} - \cdots - a_{n-1}I\right\}.$$

Example 6:

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

It has been shown that A has the characteristic equation:

$$\lambda^3 - 4\lambda^2 + 7\lambda - 10 = 0.$$

Then:

$$\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}^3 - 4\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}^2 + 7\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix} - 10\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and:

$$\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}^2 - 4\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 10A^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

or:

$$\begin{bmatrix} -5 & 2 & -16 \\ 0 & 2 & 4 \\ 5 & 0 & 10 \end{bmatrix} - 10A^{-1} = 0,$$

and finally,

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{5} & -\frac{8}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{2} & 0 & 1 \end{bmatrix}.$$

The case where $a_n = 0$ offers no exorbitant trouble, for then the matrix has no inverse.

Method VII—Frame's method (See [10].) Given matrix A, let $A_0 = I$. Using:

$$(1)\quad c_k = \frac{1}{k}\,\text{trace}\,[AA_{k-1}]$$

$$(2)\quad A_k = AA_{k-1} - c_kI,$$

we have:

$$A^{-1} = \frac{A_{n-1}}{c_n}, \qquad c_n \neq 0.$$

Example 7. Let

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

Then:

$$c_1 = \tfrac{1}{1}\,\text{trace}\,[AA_0] = 4, \qquad A_1 = AA_{k-1} - 4I = AA_0 - 4I = \begin{bmatrix} -2 & -2 & 4 \\ 2 & -1 & 2 \\ -1 & 1 & -5 \end{bmatrix},$$

$$c_2 = \tfrac{1}{2}\,\text{trace}\,[AA_1] = -7, \quad A_2 = AA_1 + 7I = \begin{bmatrix} -5 & 2 & -16 \\ 0 & 2 & 4 \\ 5 & 0 & 10 \end{bmatrix},$$

$$c_3 = \tfrac{1}{3}\,\text{trace}\,[AA_2] = \tfrac{1}{3}(-10 + 0 + 20 + 4 + 6 + 0 + 16 + 4 - 10) = 10.$$

Finally:

$$A^{-1} = \frac{A_2}{c_3} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{5} & -\frac{8}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{2} & 0 & 1 \end{bmatrix}.$$

Method VIII. Let a_k, $k = 0, 1, 2, \cdots, n-1$, be n constants. Consider the special matrix:

$$B = \begin{bmatrix} a_0 & a_{n-1} & a_{n-2} & a_{n-3} \cdots a_2 & a_1 \\ a_1 & a_0 & a_{n-1} & a_{n-2} \cdots a_3 & a_2 \\ a_2 & a_1 & a_0 & a_{n-1} \cdots a_4 & a_3 \\ \vdots & & & & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} \cdots a_1 & a_0 \end{bmatrix}.$$

Such special matrices are called circulant matrices. To invert these we use the procedure (see [13]):

1) Compute:

$$w = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}.$$

2) Calculate the eigenvalues of the given matrix by:

$$e_s(B) = \sum_{r=0}^{n-1} a_r w^{rs}, \qquad s = 0, 1, \cdots, n-1.$$

3) Calculate the inverses of these eigenvalues and denote these by e_s^{-1}.

4) Calculate the numbers:

$$b_s = \frac{1}{n}\sum_{r=0}^{n-1}(e_r^{-1})w^{-rs}, \qquad s = 0, 1, \cdots, n-1,$$

5)

$$B^{-1} = \begin{bmatrix} b_0 & b_{n-1} & \cdots & b_2 & b_1 \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ b_{n-2} & b_{n-3} & & b_0 & b_{n-1} \\ b_{n-1} & b_{n-2} & & b_1 & b_0 \end{bmatrix}.$$

Example 8. Let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ -2 & 3 & 1 \end{bmatrix},$$

which is a circulant matrix where $a_0=1$, $a_1=3$, $a_2=2$.

1) $w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

2) $e_0(A) = 6$

$$e_1(A) = 3\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) + 1 = -\frac{3}{2} + \frac{\sqrt{3}}{2}i$$

$$e_2(A) = 3\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) + 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 1 = -\frac{3}{2} - \frac{\sqrt{3}}{2}i$$

3) $e_0^{-1} = \frac{1}{6}$

$$e_1^{-1} = -\frac{1}{2} - \frac{\sqrt{3}}{6}i$$

$$e_2^{-1} = -\frac{1}{2} + \frac{\sqrt{3}}{6}i$$

4) $$b_0 = -\frac{5}{18}$$

$$b_1 = \frac{1}{3}(e_0^{-1} + e_1^{-1}w^{-1} + e_2^{-1}w^{-2}) = \frac{1}{18}$$

$$b_2 = \frac{1}{3}(e_0^{-1} + e_1^{-1}w^{-2} + e_2^{-1}w^{-4}) = \frac{7}{18}$$

5)
$$B^{-1} = \begin{bmatrix} -\frac{5}{18} & \frac{7}{18} & \frac{1}{18} \\ \frac{1}{18} & -\frac{5}{18} & \frac{7}{18} \\ \frac{7}{18} & \frac{1}{18} & -\frac{5}{18} \end{bmatrix}.$$

The method is a good one in that it works readily on matrices with complex components. If B is real and symmetrical, [13] gives simpler formulas for the computation of the eigenvalues.

Method IX—Newton's formula. In general, given a number W, suppose one wishes to calculate $1/W$. Let $x=1/W$. Then by Newton's formula, setting $f(x)=W-1/x$,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f'(x_n) \neq 0,$$

$$= x_n - \frac{W - \frac{1}{x_n}}{\frac{1}{x_n^2}}$$

$$= x_n(2 - Wx_n).$$

If we let A be a matrix whose inverse is desired, and we let X_0 be a reasonable approximation to A^{-1}, then the above formula, written in terms of matrices is:

$$X_{n+1} = X_n(2I - AX_n).$$

The following example is selected from [27]:

Example 9. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 4 \end{bmatrix}.$$

Then: choose

$$X_0 = \begin{bmatrix} -3.0 & 4.1 & -0.9 \\ 4.1 & -5.1 & 1.9 \\ -0.9 & 1.9 & -1.1 \end{bmatrix}.$$

Using the recursion formula above, it follows that:

$$X_1 = \begin{bmatrix} -4.26 & 4.14 & -0.86 \\ 4.14 & -5.06 & 1.94 \\ -0.86 & 1.94 & -1.06 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -3.9976 & 3.9864 & -1.0138 \\ 3.9864 & -4.9896 & 2.0104 \\ -1.0136 & 2.0104 & -0.9896 \end{bmatrix}.$$

The exact solution is:

$$A^{-1} = \begin{bmatrix} -4 & 4 & -1 \\ 4 & -5 & 2 \\ -1 & 2 & -1 \end{bmatrix}.$$

Concerning convergence of iteration processes applied to matrices, one should consult [22] and [17].

Method X. On certain matrices, the following method may be applied: Let $A=(I+B)$. Then, $A^{-1}=(I+B)^{-1}=I-B+B^2-B^3+\cdots$

Example 10: Let

$$A = \begin{bmatrix} 4 & 0 & 0 \\ -1 & 4 & 0 \\ 0 & -1 & 4 \end{bmatrix}.$$

Then:

$$A^{-1} = \left\{ \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \right\}^{-1}$$

$$= \tfrac{1}{4} \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -\tfrac{1}{4} & 0 & 0 \\ 0 & -\tfrac{1}{4} & 0 \end{bmatrix} \right\}^{-1}$$

$$= \tfrac{1}{4}\left\{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}0&0&0\\\tfrac{1}{4}&0&0\\0&-\tfrac{1}{4}&0\end{bmatrix} + \begin{bmatrix}0&0&0\\-\tfrac{1}{4}&0&0\\0&-\tfrac{1}{4}&0\end{bmatrix}^2 - \begin{bmatrix}0&0&0\\-\tfrac{1}{4}&0&0\\0&-\tfrac{1}{4}&0\end{bmatrix}^3 + \cdots\right\}$$

$$= \tfrac{1}{4}\left\{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}0&0&0\\-\tfrac{1}{4}&0&0\\0&-\tfrac{1}{4}&0\end{bmatrix} + \begin{bmatrix}0&0&0\\0&0&0\\\tfrac{1}{16}&0&0\end{bmatrix} - \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix} \cdots\right\}$$

$$= \tfrac{1}{4}\begin{bmatrix}1&0&0\\\tfrac{1}{4}&1&0\\\tfrac{1}{16}&\tfrac{1}{4}&1\end{bmatrix} = \begin{bmatrix}\tfrac{1}{4}&0&0\\\tfrac{1}{16}&\tfrac{1}{4}&0\\\tfrac{1}{64}&\tfrac{1}{16}&\tfrac{1}{4}\end{bmatrix}.$$

XI. Other methods and concluding remarks.

A. A newer type method which necessitates knowledge of sampling theory and game theory, concepts beyond the scope of an elementary paper, is the Monte Carlo method, developed by Von Neumann and Ulam, written up by Forsythe and Liebler in [7], and played with by Todd in [28].

B. Variations of the preceding methods are often useful for special matrices. For example, consider a matrix with the following properties:

1. It is symmetric.
2. The largest element of any row lies on the main diagonal.
3. The matrix has zeros everywhere except possibly on the main diagonal and in the positions just above and just below the main diagonal.

Such matrices occur frequently in physical problems, for example, in vibration. Preceding methods can be applied but a rapid variation is given in [24].

C. Lastly, it should be noted that if A can be written as a product of matrices whose inverses are readily found, then A^{-1} is easily produced, for:

$$\text{if:}\quad A = BCDEF, \qquad A^{-1} = F^{-1}E^{-1}D^{-1}C^{-1}B^{-1}.$$

Hence ingenuity must be used to find matrices whose inverses are simply found and whose product is A. Cholesky's method [27] is such a scheme.

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SURVEY COURSES IN MATHEMATICS FOR THE LIBERAL ARTS STUDENT*

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1. Introduction. Great interest in survey courses has been developing in the past twenty years. That educators are willing to give their students brief and general introductions into the sciences reflects one or both of two trends. First, there is growing adherence to the belief that what is called by philosophers "the unity of science" is more than a fiction. Second, this willingness indicates that knowledge of the sciences, however small, somehow furnishes the individual with a capacity to make experience more meaningful and positive. The growing interest in science fiction is perhaps a popular indication of man's acceptance of this explanation.

Mathematics has played so important a role in recent scientific advances that it becomes necessary to consider what methods are most effective in furnishing the student with enough information about it and the part that it plays in our progress. Our discussion has to do with the purposes of survey courses in mathematics, with types of courses which can be designed to fulfill these purposes, and with suggestions for available textbook material to meet the needs of the courses described.

2. Needs and purposes. Among the most obvious purposes of survey courses in general is what a mathematician might call "communication closure." The abundance of scientific terminology in everyday speech makes knowledge of such terminology mandatory. Knowledge of terminology alone is not enough to provide the desired closure. An acquaintance with the ideas for which the terms stand is also necessary. Many terms and ideas have applications outside of mathematics, as well as within, and it is understanding these that is specially important. The central mathematical notion that carries scientific information is the notion of function. A study of functions and their properties can involve mathematics which is well within the grasp of industrious and imaginative college students.

A second purpose for a survey course in mathematics which is closely allied to the previous one is merely to teach the student the meaning of numerical statements as they are used in communicating scientific data. This would require little investigation into basic mathematics. It would require mainly the numerical techniques (such as those of statistics) by which scientific data may be evaluated and disseminated.

A survey course in mathematics can also serve the purpose of furnishing the student with a solid knowledge of ideas about truth and validity. The advantage of teaching logic from a mathematical point of view, rather than from the classical (or philosophical), is the familiarity which such a treatment affords with uses of

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formal language. This familiarity with formal language in turn furnishes a device through which the student may learn more about the needs and uses of theory and generalization.

The morning session of the meeting of the Association in Baltimore, December 31, 1953, discussed another important purpose of mathematics surveys, and described ably the steps being taken to fulfill the purpose. I refer now to the growing importance to social scientists of mathematical techniques in constructing models. Because the matter has already been covered in Baltimore as well as it can be, I shall not discuss it further in what follows.

The student enrolled in a science curriculum furnishes an example of a class of prospective customers for the survey course in mathematics. Such a student is usually required to take courses in techniques, but most schools fail even to offer him the opportunity to study fundamental ideas which might equip him with more thorough understanding of mathematics and the sciences, and their relation to one another. If such an opportunity is presented it is usually not until the student is doing graduate work. If the student is in the liberal arts or if he plans to teach science or mathematics before doing graduate work, it is especially important for him to be acquainted with the basic notions with which a good survey course can supply him. It is contended that little time is left in the college career of such a student for "culture" courses like these, but it is certainly true that to offer a course in basic mathematics (and basic science) would be a gallant step in the right direction, even if such a course were elected rather than required.

3. Techniques. Before discussing particular suggestions for survey course topics, it seems a good idea to list several pedagogical techniques that such courses indicate as desirable. A survey course is best devised when it presents either new material, or digests and recasts old material in a new form. To go over high school material slowly leaves most students indifferent. To do this quickly makes the course more difficult than necessary and frustrates the student. Even with novel approaches (or perhaps even because of them) it is not necessary to require the student to have any previous background in mathematics. But if this is so, then what is required is a close touch throughout the course with an intuition which has been educated by the instructor early in the course. One good way of achieving these ends is through a detailed study of both the intuitive and the formal development of the real number system. Applying such techniques will do much to get the student's interest to participate in the study and to do away with the stigma which surrounds studies in mathematics.

Other suggestions are of a more picayune nature. The student will profit by reading simple mathematics on his own and reporting on some topic not discussed in the classroom. Though many of the topics suggested below are of such a nature that they can best be presented by lecture, chances do arise for spirited

classroom discussion about issues that arise out of the student's existing misconceptions, or out of consideration of philosophical questions concerning mathematics and science. Such discussions can be stimulated. Of course meaningful exercises that are not too difficult are always instrumental in putting across a point—a technique or an idea.

4. Course contents. An intensive one-year survey of mathematics is effective in attaining both proficiency in technique, and understanding of basic ideas. There is time to discuss the ideas which underlie analysis—the real numbers, algebra, and the theory of real functions, as well as related ideas from applications of analysis to measurement in general, to statistics, and to other fields to which application is immediate. There is a chance to illustrate the ideas with examples and techniques which are more than elementary, so that a follow-up course in technical calculus can make the student proficient in the uses of analysis. Finally, through topics from mathematical logic, the student can be given an insight into axiomatics and become acquainted with the nature of formalism and generality.

There are two textbooks in particular which follow the suggested lines. [7] is a three volume paper-bound text used at the University of Chicago and written by the department there, and represents an early attempt at developing this type of course. [1] is a new, and, in my opinion, excellent book for such a course. It is available in an offset edition only at present and has been used with success at the University of Washington.*

We have been working at Penn State on a similar type of course for the past two and a half years. Our course differs from the one above in these respects: first, it is a one semester course; second, because of this, certain topics included in the previously described course are not included in ours. The most notable omission is that no attempt is made at teaching involved techniques with complicated functions. Polynomials are used for illustrations and examples throughout the course. Ideas from mathematical logic are distributed throughout the course, rather than introduced as a separate topic. The same is true of applications of analysis, which are confined largely to explanations of simple examples from physics.

What remains, then, is an intensive discussion of the real number system, and a thorough discussion of the idea of a function, or the properties of functions and their use in social and natural science. The course closes with as long a discussion of axiomatics and formalism as time allows. Two books have been used so far: [4] and [2]. Both books have been incomplete for the purposes of our course, and supplementary reading, especially of material on the real number system, has had to be given.

There are a number of acceptable alternative choice of topics for survey courses in mathematics. Certainly study of basic notions of analysis does not

* *Editorial note.* The printed edition has just been published.

exhaust all the possibilities. Let us look briefly at several other ideas.

A course that attempts to fulfill the communication closure mentioned earlier can be oriented from the point of view of numerical methods. The only techniques which have to be presented thoroughly are those of arithmetic. Other topics, such as applications, can be adequately explained through the notion of formula. The idea of function, though fruitful, need not be mentioned at all. Techniques of measurement and approximation can be given purely operationally, without the need for strict development of algebra, geometry, and calculus. A course like this leaves much to be desired but has the advantage that it can be presented to college students who have had no mathematics in high school and who are less mature thinkers than we would desire.

A course in foundations through mathematical logic has much to recommend it, being an avenue into simple abstract thinking and idea construction. This course would spend time on the predicate calculus, presumably, but would emphasize proof theory, the idea of axiomatization, and the notions which usually go under the title of metamathematics. Textbook suggestions for such a course include [6] and [5]. For the advanced student, [8] is a suggestion, but the first two books share the advantage of having a symbolism which is more immediate.

A final suggestion for those who believe that a college student in liberal arts should be able to pass a mathematics course at all costs—that he should be "exposed" to mathematics. A survey of "game" topics in mathematics can be developed that will activate the interest of the student along mathematical lines. A course like this could contain some group theory, combinatorial topology, decidable games like "nim," magic squares, and so forth. Of course, such content is very difficult to motivate, and such a course would be difficult to integrate and unify.

One factor to keep in mind in formulating a course is the interest of the student, without which many difficulties arise. Motivating vividly, educating the intuition, and keeping as close as possible to experience are effective influences in fostering this interest.

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THE POWER SERIES COEFFICIENTS OF $\zeta(s)$*

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Let $\zeta(s)$ be the Riemann zeta function defined by

$$\zeta(s) = \sum_1^\infty n^{-s} \qquad R(s) > 1.$$

The function can be analytically continued for $R(s)>0$ by [1, Satz 440]

$$g(s) = (2^{1-s} - 1)\zeta(s) = \sum_1^\infty \frac{(-1)^n}{n^s}. \tag{1}$$

Well known is $\lim_{s\to 1} (s-1)\zeta(s) = 1$, so that $\zeta(s)$ can be written as

$$\zeta(s) = \frac{1}{s-1} + \sum_0^\infty A_n(s-1)^n. \tag{2}$$

The purpose of this paper is to investigate these coefficients.

From (1) we obtain, for k any non-negative integer,

$$g^{(k)}(s) = (-1)^k \sum_1^\infty \frac{(-1)^n \log^k n}{n^s}. \tag{3}$$

To get a power series for $g(s)$, write

$$2^{1-s} - 1 = e^{(1-s)\log 2} - 1 = \sum_1^\infty \frac{(-1)^n \log^n 2}{n!}(s-1)^n,$$

and multiply by the series in (2) to get

$$g(s) = \sum_{k=0}^\infty \sum_{t=1}^{k+1} \frac{(-1)^t \log^t 2}{t!} A_{k-t}(s-1)^k, \qquad A_{-1} = 1.$$

Hence

$$g^{(k)}(1) = k! \sum_{t=1}^{k+1} \frac{(-1)^t \log^t 2}{t!} A_{k-t}. \tag{4}$$

From a simple geometric argument follows

$$\sum_{n=1}^x \frac{\log^k n}{n} = \frac{\log^{k+1} x}{k+1} + \gamma_k + o(1), \tag{5}$$

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and hence

$$\sum_{n=1}^{2x} \frac{\log^k n}{n} = \frac{(\log 2x)^{k+1}}{k+1} + \gamma_k + o(1). \tag{6}$$

Also

$$\begin{aligned} 2\sum_{n=1}^{x} \frac{(\log 2n)^k}{2n} &= \sum_{n=1}^{x} \frac{1}{n} \sum_{t=0}^{k} \binom{k}{t} (\log^{k-t} 2)(\log^t n) \\ &= \sum_{t=0}^{k} \binom{k}{t} \log^{k-t} 2 \sum_{n=1}^{x} \frac{\log^t n}{n} \\ &= \sum_{t=0}^{k} \binom{k}{t} \log^{k-t} 2 \left[\frac{\log^{t+1} x}{t+1} + \gamma_t + o(1)\right] \\ &= \frac{(\log 2x)^{k+1}}{k+1} - \frac{\log^{k+1} 2}{k+1} + \sum_{t=0}^{k} \gamma_t \binom{k}{t} \log^{k-t} 2 + o(1). \end{aligned} \tag{7}$$

Subtracting (7) from (6) gives

$$\sum_{n=1}^{2x} \frac{(-1)^{n-1} \log^k n}{n} = \frac{\log^{k+1} 2}{k+1} - \sum_{t=0}^{k-1} \gamma_t \binom{k}{t} \log^{k-t} 2 + o(1),$$

and we get, by letting $x \to \infty$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \log^k n}{n} = \frac{\log^{k+1} 2}{k+1} - \sum_{t=0}^{k} \gamma_{k-t} \binom{k}{t} \log^t 2. \tag{8}$$

From (3), (4), and (8) follows

$$k! \sum_{t=0}^{k+1} \frac{(-1)^t \log^t 2}{t!} A_{k-t} = (-1)^{k-1} \left[\frac{\log^{k+1} 2}{k+1} - \sum_{t=1}^{k} \gamma_{k-t} \binom{k}{t} \log^t 2\right],$$

and from this we easily get the

THEOREM. *If*

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} A_n (s-1)^n,$$

then

$$A_n = \frac{(-1)^n \gamma_n}{n!},$$

where γ_n is defined in (5).

We give a second proof of the theorem. Write [2, p. 14]

$$\zeta(s) - \frac{s}{s-1} = \zeta(s) - \frac{1}{s-1} - 1 = h(s) = \sum_{n=0}^{\infty} B_n(s-1)^n$$
$$= s\int_1^{\infty} \frac{[x]-x}{x^{s+1}}\,dx \qquad R(s) > 0,$$

where $B_n = A_n$, unless $n=0$, where $B_0 = A_0 - 1$. Hence

$$h^{(k)}(1) = k!B_k = (-1)^k\left[\int_1^{\infty} \frac{[x]-x}{x^2}\log^k x\,dx - k\int_1^{\infty} \frac{[x]-x}{x^2}\log^{k-1} x\,dx\right]$$

(9)

$$= (-1)^k \lim_{N\to\infty}\left\{\sum_{t=1}^{N-1}\left[t\int_t^{t+1} \frac{\log^k x - k\log^{k-1} x}{x^2}\,dx\right.\right.$$
$$\left.\left. - \int_1^N \frac{\log^k x - k\log^{k-1} x}{x}\,dx\right\}\right.$$

It is easily seen that

$$\int_t^{t+1} \frac{\log^k x - k\log^{k-1} x}{x^2}\,dx = -\frac{\log^k x}{x}\Big|_t^{t+1} = -\frac{\log^k(t+1)}{t+1} + \frac{\log^k t}{t}.$$

Therefore

$$\sum_{t=1}^{N-1} t\int_t^{t+1} \frac{\log^k x - k\log^{k-1} x}{x^2}\,dx = \sum_{t=1}^{N-1} t\left[\frac{\log^k t}{t} - \frac{\log^k(t+1)}{t+1}\right]$$
$$= \sum_{t=1}^{N-1} \frac{\log^k t}{t}[t-(t-1)] - \frac{N-1}{N}\log^k N$$
$$= \sum_{t=1}^{N-1} \frac{\log^k t}{t} - \frac{N-1}{N}\log^k N.$$

$$\int_1^N \frac{\log^k x - k\log^{k-1} x}{x}\,dx = \frac{\log^{k+1} N}{k+1} - \log^k N.$$

Therefore, for $k>0$,

$$k!B_k = (-1)^k \lim_{N\to\infty}\left[\sum_{t=1}^{N-1} \frac{\log^k t}{t} - \left(1-\frac{1}{N}\right)\log^k N - \frac{\log^{k+1} N}{k+1} + \log^k N\right]$$
$$= (-1)^k \lim_{N\to\infty}\left[\sum_{t=1}^{N} \frac{\log^k t}{t} - \frac{\log^{k+1} N}{k+1}\right] = (-1)^k\gamma_k,$$

where the limit is evaluated by (5). For $k=0$ the integral in (9) is similarly evaluated to give $B_0 = \gamma_0 - 1$.

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THE NUMBER OF VERTICES OF A POLYHEDRON

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1. Introduction. Consider the following (linear programming) problem: Find values of $x_1, \cdots, x_n$ which maximize (minimize) the linear form

(1) $$x_1c_1 + \cdots + x_nc_n \quad \text{subject to the conditions that}$$

(2) $$x_j \geqq 0 \ (j = 1, \cdots, n) \quad \text{and} \quad \sum_{j=1}^{n} a_{ij}x_j \geqq b_i \ (i = 1, \cdots, m)$$

where the a_{ij}, b_j, c_j are constants.

In general, a set of linear inequalities defines a convex set in affine space; n-dimensional affine space A_n is the set of points $(c_1, c_2, \cdots, c_n)$ where c_1, $c_2, \cdots, c_n$ are real numbers. A set is convex in affine space if it contains the segment joining any two of its points. The convex set defined by (2) is the intersection of half spaces. It has vertices, edges, *etc.* Geometrically, a solution to the above problem is a point of the convex set defined by (2) which also maximizes (minimizes) (1). This point is generally a vertex of the boundary. For simplicity assume that (1) and (2) are normalized. Then this is the vertex $(x_1^0, x_2^0, \cdots, x_n^0)$ for which (1) assumes its maximum (or minimum) distance from the origin.

Briefly, the "simplex process" is one of several iterative procedures used in solving this problem. The iterations of this process consist of translating the hyperplane corresponding to (1) in a parallel direction each time evaluating its distance from the origin at the intersected vertex of the convex set defined by the inequalities. The process is repeated in such a way that the translated hyperplane in each step yields improved results leading to the optimal distance. The simplex process also involves a useful criterion which eliminates some of the vertices of the convex set as possible trial points [3].

The number of iterations involved is decided for each problem separately. Unfortunately, so far, this number is only known to be dominated by $C_{n+m,n}$ and as will be seen below this is an unsatisfactory estimate of the number of vertices of the convex sets of this problem. One would like an upper bound to the number of trial vertices knowing the number of inequalities and variables involved (the number of variables determines n, the dimension of the space). Hence the upper bound we wish to study here is of the form:

(number of vertices)$_n \leqq$ (a function of the number of inequalities)$_n$, n being the dimension of the space. Since each inequality defines at most a half-space with a hyperplane of dimension $(n-1)$ as boundary, it suffices to study this problem in general in the form: (number of vertices)$_n \leqq$ (a function of the number of $(n-1)$ dimensional hyperplanes)$_n$.

A rough upper bound which does not always take into consideration the polyhedral [1] property of the problem is available. If we denote by $F_i (i = 0, 1, \cdots, n-1)$ the number of ith dimensional faces of a convex polyhedron in n-dimensional affine space then we have,

$$F_0 \leqq \frac{F_{n-1}!}{(F_{n-1}-n)!\,n!}, \tag{3}$$

which is obtained by intersecting the $(n-1)$ dimensional hyperplanes n at a time.

Digressing for a moment, let us consider the tetrahedron in n-dimensional space (the n-dimensional simplex). The number of its kth dimensional faces F_k is given by

$$F_k = \frac{(n+1)n(n-1)\cdots(n-k+1)}{(k+1)!}, \qquad (k = 0, 1, \cdots, n). \tag{4}$$

Now it is a known fact (proof by induction) that the simplex is a minimal polyhedron, for any value of the dimension n, in that it has the least number of faces $F_i (i=0, \cdots, n-1)$ of any polyhedron of that dimension.

Remark 1: Since $F_0 = n+1$, and $F_{n-1} = n+1$, for the simplex, the latter being a minimal polyhedron, a desirable upper bound for F_0 as a function of F_{n-1} would yield equality on substituting $(n+1)$ for F_{n-1}. Now (3) satisfies this property.

The object here is to improve on (3) by a closer study thus obtaining a function $f(F_{n-1})$ with the property that

$$F_0 \leqq f(F_{n-1}) \leqq \frac{F_{n-1}!}{(F_{n-1}-n)!\,n!}. \tag{5}$$

(Note that the second equality cannot be avoided because of the above remark.)

2. Solution. The Euler-Descartes formula

$$\sum_{i=0}^{n-1} (-1)^i F_i = \begin{matrix}(0,\ n \text{ even})\\ (2,\ n \text{ odd})\end{matrix} \tag{6}$$

holds for polyhedra in n-dimensions that are homeomorphic images of the simplex in that dimension. We need not go into details here. It suffices to point out that the convex sets involved in our problem satisfy (6) or can be made to do so on minor changes. Thus the inequalities in (2) define a convex set, which is a polyhedron, perhaps unbounded, in n-space with a boundary of at most $m+n$ hyperplanes. Hence for our purpose $F_{n-1} = m+n$.

An easy proof of (6) for the simplex is obtained by considering the n-dimensional simplex which according to (4) has $(n+1)$ vertices. By combining these vertices two at a time, three at a time, *etc.*, the number of edges, planes, *etc.*, of the simplex is obtained. If grouped as in (6), the quantities on the right are obtained [1].

The following relations, also used here, are:

$$nF_0 \leqq 2F_1 \tag{7}$$

(*e.g.*, for $n=3$ at least 3 lines meet in a vertex with each line joining two vertices),

$$nF_{n-1} \leqq 2F_{n-2}, \tag{8}$$

$$F_k \leqq \frac{F_{n-1}!}{(F_{n-1}-n+k)!(n-k)!}. \tag{9}$$

(This relation suffers from similar defects mentioned above.)

$$F_k \geqq \frac{(n+1)n(n-1)\cdots(n-k+1)}{(k+1)!}. \tag{10}$$

(See (4) and the statement following it.) Note that (7) and (9) yield

$$\frac{n}{2}F_0 \leqq F_1 \leqq \frac{F_{n-1}!}{(F_{n-1}-n+1)!(n-1)!}. \tag{11}$$

Now

$$\frac{2}{n}\,\frac{F_{n-1}!}{(F_{n-1}-n+1)!(n-1)!} \leqq \frac{F_{n-1}!}{(F_{n-1}-n)!n!} \tag{12}$$

since $F_{n-1} \geqq n+1$ with strict inequality implied if $F_{n-1} > n+1$. Thus the left hand side of (12) is a better upper bound for F_0 than (3).

In (6) and (7) let $n=3$. Multiply (6) by (-1) and substitute for F_1 in (6) the majorized quantity $\frac{3}{2}F_0$ and the following is obtained:

$$F_0 \leqq 2F_2 - 4 \tag{13}$$

which for $F_2 \geqq 4$ is dominated by the right side of (3) with $n=3$.

Now let $n=3$ and $F_2=10$ be assumed, then (3) yields $F_0 \leqq 10!/7!3!=120$ while from (13) we have $F_0 \leqq 16$ (at most 16 iterations as compared to 120).

Let $n=4$ in (6), (7) and (9). We have from (9)

$$F_2 \leqq \tfrac{1}{2}F_3(F_3-1). \tag{14}$$

Substituting from (7) in (6), the following result is obtained

$$F_0 - 2F_0 - F_2 - F_3 \geqq 0 \quad \text{or} \quad F_0 \leqq F_2 - F_3. \tag{15}$$

Instead of F_2 we substitute the majorizing quantity from (14) and simplify. Thus (15) becomes

$$F_0 \leqq \frac{F_3}{2}(F_3-3), \tag{16}$$

which for $F_3 \geqq 5$ is dominated by the right side of (3) with $n=4$. Here if we assume $F_3=10$, (16) yields 35 as an upper bound whereas (3) gives 210. Note that (5) is satisfied in both cases.

Generalization.

A. n even:

We shall first state the result and then indicate how it is obtained:

$$F_0 \leqq \frac{2}{n-2}\left\{\sum_{p=1}^{n-2/2}\binom{F_{n-1}}{n-2p} - F_{n-1} - 2^n + \left[2 + \frac{n(n+3)}{2}\right]\right\}. \tag{17}$$

Since n is even (6) is given by

$$F_0 - F_1 + F_2 - F_3 + F_4 - \cdots + F_{n-2} - F_{n-1} = 0. \tag{18}$$

Using (17) we have

$$F_0 - \frac{n}{2}F_0 + F_2 - F_3 + F_4 - \cdots + F_{n-2} - F_{n-1} \geqq 0 \tag{19}$$

or

$$\frac{n-2}{2}F_0 - F_2 + F_3 - F_4 + \cdots - F_{n-2} + F_{n-1} \leqq 0. \tag{20}$$

Substitute for the F_i with positive signs (except F_0 and F_{n-1}) majorized quantities from (10) and for those with negative signs majorized quantities from (9) after multiplying the latter by (-1). The sum of the quantities obtained from the first substitution is the negative of the last two expressions to the right of (17) having made use of the expansion of $(1+1)^n-(1-1)^n$. The second substitution yields the negative of the first expression to the right of (17). By transposing $+F_{n-1}$ to the right, the second term on the right of (17) is obtained.

B. n odd:

Do the same as above with the exception that no substitution is made for F_{n-2}. The reason for this is that when transposed to the right it appears with a negative sign and from (8) it can be majorized by $\frac{1}{2}nF_{n-1}$ which is a better substitution. In the first of the two expressions below this exception is not made, but it is in the second:

$$F_0 \leqq \frac{2}{n-2}\left\{\sum_{p=1}^{n-3/2}\binom{F_{n-1}}{n-2p} + F_{n-1} - 2^n + \frac{(n-1)n}{2}\right\} \tag{21}$$

$$F_0 \leqq \frac{2}{n-2}\left\{\sum_{p=1}^{n-3/2}\binom{F_{n-1}}{n-2p} - \frac{n-2}{2}F_{n-1} - 2^n + (n-1)n\right\}. \tag{22}$$

That remark (1) is true in each case can be easily verified.

Remark 2: It is not difficult to prove that (17) and (22) are dominated by the right side of (3) for $F_{n-1} \geqq 2n$.

Remark 3: It is clear that a value of F_{n-1} as a function of n exists such that

$$\frac{2(F_{n-1}!)}{(F_{n-1}-n+1)!n!}$$

majorizes (17) and (22) for all values of F_{n-1} greater than or equal to this value.

THEOREM 1. *There exists no upper bound on F_0 of the form $aF_{n-1}+b$, with a and b as functions of n, satisfying*

$$F_0 \leqq aF_{n-1}+b \leqq 2\cdot\frac{F_{n-1}!}{(F_{n-1}-n+1)!n!} \tag{23}$$

for all $F_{n-1}\geqq n+1$ and for $n=4$, or $n\geqq 7$.

Proof. Since the two extremes in (23) yield equality for the simplex, we must have:

$$n+1=(n+1)a+b. \tag{24}$$

Suppose $b=0$; then $a=1$ and it is clear that $F_0\leqq F_{n-1}$ is not true in general (*e.g.*, the hypercube has $F_0=2^n$ and $F_{n-1}=2n$). Thus

$$b=(n+1)-(n+1)a. \tag{25}$$

It is possible to obtain a convex n-dimensional polyhedron with $F_{n-1}=n+2$ by cutting the (n) simplex with an $(n-1)$ dimensional hyperplane. Hence from (23) and (25) the following must hold:

$$(n+2)a+[(n+1)-(n+1)a]\leqq 2\frac{(n+2)!}{3!n!}=\frac{(n+2)(n+1)}{3} \tag{26}$$

or

$$a\leqq\frac{(n+1)(n-1)}{3}. \tag{27}$$

Using the equality sign in (27) it will be seen that even with this value of a, the assumed upper bound does not hold. Thus for the hypercube one must have

$$2^n\leqq\frac{(n+1)(n-1)}{3}\cdot 2n+(n+1)-\frac{(n+1)^2(n-1)}{3} \tag{28}$$

which is false for $n\geqq 7$.

The proof for $n=4$ follows since one of the regular polyhedra in 4 dimensions has $F_0=600$ and $F_3=120$.

At the outset it was our hope to prove that an upper bound as a linear function of F_{n-1} with coefficients as a function of n existed.

The results obtained here, though more complicated than the fundamental upper bound mentioned early in the paper, comprise a first step in producing a

finer upper bound [2]. It is possible that a generalization of the above theorem would show that finer upper bounds than those obtained here do not exist.

These improved results, as has been shown in the cases of $n=3$ and $n=4$, are valuable in estimating the number of iterations and hence the time that an electronic computor would require to solve the problem.

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NOTE ON A HEAT CONDUCTION PROBLEM

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1. Introduction. Rainville [4] using a new method solved the problem of the heat conduction in a wedge of angle β when the sides are maintained at temperature zero and the initial temperature was unity. The solution in a different form was in effect obtained by Carslaw [1] and the result has been quoted by Jaegar [3], Carslaw and Jaegar [2], and Sowerby and Cooke [5]. The purpose of this note is to give a method of obtaining the result by the use of Integral Transforms. The procedure is formal but it illustrates the great utility of the Integral Transform method of *obtaining a result*, and, as Tranter [6] pointed out, it reduces the procedure almost to a "drill." Follow the correct procedure and the result comes out automatically. The great advantage is that *no assumption of the form of the solution need be made.*

2. The problem. We have to solve the equation

$$\kappa^{-1}u_t = u_{rr} + (1/r)u_r + (1/r^2)u_{\theta\theta} \tag{1}$$

where subscripts denote partial differentiation, u is the temperature, κ the diffusivity and r and θ are cylindrical coordinates. The boundary conditions are:

$$u \to 1 \quad \text{as} \quad t \to 0+ \quad \text{for} \quad 0 < r, \quad 0 < \theta < \beta,$$
$$u \to 0 \quad \text{as} \quad \theta \to 0+ \quad \text{for} \quad 0 < t, \quad 0 < r,$$
$$u \to 0 \quad \text{as} \quad \theta \to \beta- \quad \text{for} \quad 0 < t, \quad 0 < r.$$

For convenience we shall write $u=1-w$ so that the boundary conditions are, shortly,

$$w = 0 \quad \text{when} \quad t = 0,$$
$$w = 1 \quad \text{when} \quad \theta = 0 \quad \text{or} \quad \theta = \beta.$$

3. Integral Transforms. We shall use the finite Fourier sine transform defined by

$$\bar{f}(p) = \int_0^\pi f(\phi) \sin p\phi d\phi$$

where p is integral. The inverse formula is

$$f(\phi) = (2/\pi) \sum_{p=0}^{\infty} \bar{f}(p) \sin \phi p. \tag{2}$$

We shall also use the Hankel Transform defined by

$$\bar{f}(p) = \int_0^\infty rf(r)J_s(pr)dr$$

whose inverse is

$$f(r) = \int_0^\infty p\bar{f}(p)J_s(rp)dp. \tag{3}$$

In our case we shall denote the finite sine transform of w by $\bar{w}$, and the Hankel transform of $\bar{w}$ by $\bar{w}'$ following the notation of Tranter [6].

4. The solution. In equation (1) we write $\theta = \beta\phi/\pi$ and so the equation to solve is

$$\kappa^{-1}w_t = w_{rr} + (1/r)w_r + (\pi^2/r^2\beta^2)w_{\phi\phi} \tag{4}$$

with boundary conditions

$$w = 0 \quad \text{when} \quad t = 0, \tag{5}$$

$$w = 1 \quad \text{when} \quad \phi = 0 \quad \text{or} \quad \phi = \pi. \tag{6}$$

Multiply equation (4) by $\sin p\phi$ and integrate between 0 and π. Performing a double integration by parts on the last term we obtain, observing that w_ϕ must be finite and taking account of the boundary conditions (6),

$$\kappa^{-1}\bar{w}_t = \bar{w}_{rr} + (1/r)\bar{w}_r - \bar{w}\pi^2p^2/\beta^2r^2 + (\pi^2p/\beta^2r^2)\{1 - (-1)^p\}.$$

We shall see later that p must be an odd integer and so this equation reduces to

$$\kappa^{-1}\bar{w}_t = \bar{w}_{rr} + (1/r)\,\bar{w}_r - s^2\bar{w}/r^2 + 2\pi s/\beta r^2 \tag{7}$$

on writing $s = \pi p/\beta$.

We now multiply equation (7) by $rJ_s(qr)$ and integrate to obtain the Hankel transform. Two integrations by parts are again required. Omitting the details of the integration, which are given by Tranter and are straightforward, we obtain

$$\kappa^{-1}\bar{w}_t' = -q^2\bar{w}' + (2\pi s/\beta)\int_0^\infty r^{-1}J_s(qr)dr. \tag{8}$$

Now (Watson [7a]) the integral in this equation is equal to $1/s$. Hence

$$\kappa^{-1}\bar{w}_t' = -q^2\bar{w}' + 2\pi/\beta. \tag{9}$$

We now solve this equation in the one independent variable t, taking account of the boundary condition (5) which gives $\bar{w}'=0$ when $t=0$. We obtain

$$\bar{w}' = (2\pi/\beta q^2)(1 - e^{-q^2t\kappa}).$$

On inverting this by equation (3) we obtain

$$\bar{w} = (2\pi/\beta)\int_0^\infty q^{-1}J_s(qr)dq - (2\pi/\beta)\int_0^\infty e^{-q^2t\kappa}q^{-1}J_s(qr)dq$$

or

$$\bar{w} = (2/p) - (2\pi/\beta)I_s,$$

writing

$$I_s = \int_0^\infty e^{-q^2t\kappa}q^{-1}J_s(qr)dq.$$

Inverting again by equation (2) we obtain

$$w = \frac{4}{\pi}\sum_p \frac{\sin p\phi}{p} - \frac{4}{\beta}\sum_p I_s \sin p\phi.$$

Remembering that p is odd, say $p=2n+1$, we thus finally obtain on summing the sine series

$$w = 1 - (4/\beta)\sum_{n=0}^{\infty} \sin k\theta \int_0^\infty e^{-q^2t\kappa}J_k(qr)q^{-1}dq,$$

where $k=(2n+1)\pi/\beta$.

This gives

$$u = \frac{4}{\beta}\sum_{n=0}^{\infty} \sin k\theta \int_0^\infty \frac{e^{-q^2\kappa t}J_k(qr)}{q}\,dq,$$

which reduces to the form given by Rainville on expressing the integral in terms of hypergeometric functions. See Watson [7b]. (There are two slight errors of detail in Rainville's result. The power of $\rho^2/4h^2t$ in his expression for $G_n(\rho)$ should be $(2n+1)\pi/2\beta$ instead of $(2n+1)\pi/\beta$, and the term $2n+1$ appearing in his formula for a_n should be in the denominator instead of the numerator.)

Note 1. For even values of p the last term of equation (7) vanishes and so

there is no last term in equation (9). It is then not possible to solve this equation and satisfy the boundary condition $\bar{w}' = 0$ when $t = 0$. Hence p must be odd.

Note 2. Equation (8) was obtained on the assumption that $[r\bar{w}J_s'(qr)]_0^\infty$ and $[r\bar{w}_r J_s(qr)]_0^\infty$ vanish at both limits. This assumption must be verified *a posteriori*. Alternatively since the method aims at obtaining a *result*, and is purely formal, the result must be justified directly in any case. This can easily be done.

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ON QUASI-IDEMPOTENT MATRICES*

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Introduction. Let A be a square matrix over the field of complex numbers C, and let $F(x)$ be a polynomial matrix of the same order over the ring $C[x]$ of polynomials in the indeterminate x with complex coefficients. If

$$A^r = F(r)$$

for all positive integers r, we will say that A is *quasi-idempotent* and that $F(x)$ is an *exponential polynomial matrix* associated with A. It is clear that any idempotent matrix P is a quasi-idempotent matrix associated with the polynomial matrix P. An example of a quasi-idempotent matrix A which is not idempotent is given by the A and the $F(x)$ displayed below:

$$A = \begin{pmatrix} -9 & 48 & -18 \\ -1 & 5 & -2 \\ 3 & -16 & 6 \end{pmatrix}; \quad F(x) = \begin{pmatrix} -12x+3 & 48x & -24x+6 \\ -x & 4x+1 & -2x \\ 4x-1 & -16x & 8x-2 \end{pmatrix}.$$

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Other examples of such matrices are found in the literature [2, 3, 4], although this property does not seem to have been studied explicitly. In this paper, a miniature theory of quasi-idempotent matrices is developed by elementary methods.

An examination of the example above is illuminating in predicting what may be expected of quasi-idempotent matrices in general. It is easy to verify that $F(1/2)$ is a square root of A and it is indeed true that $F(1/n)$ is an n-th root of A for any positive integer n. Moreover, A does not have an inverse in the ordinary sense (the third column is the double of the first) and $F(0)$ is not the identity matrix, E. Nevertheless, it may be verified by direct computation that:

$$[F(0)]^2 = F(0), \qquad F(0)A = AF(0) = A, \quad \text{and} \quad F(-1)A = AF(-1) = F(0).$$

This means that A and $F(-1)$ generate a multiplicative group with identity $F(0)$ and that $F(-1)$ is the inverse of A in this group. In the sequel, if A is a matrix and B is an idempotent matrix such that $B_0A = AB_0 = A$, we will say that B_0 *is an identity for* A.

The arguments in Section 1 depend on the natural isomorphism [1, 5] between the ring $(C[x])_n$ of square polynomial matrices of order n over the ring $C[x]$ of polynomials in the indeterminate x with coefficients in C, and the ring $C_n[X]$ of matric polynomials in the indeterminate X with coefficients in the ring C_n of square matrices of order n over C. In this isomorphism, if $F(x) \in (C[x])_n$ and the element in the i-th row and j-th column of $F(x)$ is $\sum_k a_{ij,k}x^k$, then the image $F^*(X)$ is defined to be $\sum_k A_kX^k$, where A_k is the matrix with $a_{ij,k}$ in the i-th row and j-th column. In the arguments below, $F(x)$ and $F^*(X)$ will be used to designate an image pair in this isomorphism. A similar notation will be used in the similar isomorphism between $(C[x, y])_n$ and $C_n[X, Y]$.

If the matric polynomial $F^*(X)$ is the image of the polynomial matrix $F(x)$ in the natural isomorphism and r is a positive integer, then $F^*(rE) = F(r)$. It follows that a matrix A is quasi-idempotent if and only if there is a matric polynomial $F^*(X)$ such that $A^r = F^*(rE)$ for all positive integers r. Such a matric polynomial will be called an exponential matric polynomial and the original problem will be studied in this form.

1. The characterization of exponential matrix polynomials. A matric polynomial $F^*(X)$ is an exponential matric polynomial if and only if $[F^*(E)]^r = F^*(rE)$ for all positive integers r. A natural direct argument extends this to the following:

THEOREM 1. *A matric polynomial $F^*(X)$ is an exponential matric polynomial if and only if $F^*(rE)F^*(sE) = F^*(rE+sE)$ for all pairs of positive integers r and s.*

Now let $F^*(X)$ be an exponential matric polynomial. The fact that scalar matrices commute with all matrices implies that $F^*(rE)F^*(sH)$ is equal to $[F^*(X)F^*(Y)]_{X=rE, Y=sE}$. That is, if $F^*(X)$ satisfies the condition in Theorem 1, then the matric polynomial $F^*(X)F^*(Y)$ takes on the same value as the matric

polynomial $F^*(X+Y)$ for each pair rE, sE, where r and s are any positive integers. This in turn implies that the polynomial matrix $F(x)F(y)$ takes on the same values as the polynomial matrix $F(x+y)$ for all pairs of positive integers r and s. Hence, the polynomial in the i-th row and j-th column of $F(x)F(y)$ takes on the same values as the polynomial in the i-th row and j-th column of $F(x+y)$ for all pairs of positive integers r, s. For each i and j, these polynomials are equal and $F(x)F(y)=F(x+y)$. Finally, it follows from the isomorphism that $F^*(X)F^*(Y)=F^*(X+Y)$. Since the converse implication is clear, we have:

THEOREM 2. *A matric polynomial $F^*(X)$ is an exponential matric polynomial if and only if $F^*(X)F^*(Y)=F^*(X+Y)$.*

Comparing coefficients results in:

THEOREM 3. *If $F^*(X)=B_0+B_1X+B_2X^2+\cdots+B_nX^n$, $B_n\neq 0$, then $F^*(X)$ is an exponential matric polynomial if and only if*

$$B_iB_j = B_{i+j}\binom{i+j}{i}, \qquad \text{where } B_t = 0 \text{ for } t > n.$$

2. The characterization of quasi-idempotent matrices. If $F^*(X)$ and $G^*(X)$ are matric polynomials and $F^*(rE)=G^*(rE)$ for all positive integers r, then $F^*(X)=G^*(X)$. Thus if A is quasi-idempotent, there is a unique exponential matric polynomial associated with A. Let A be quasi-idempotent and let

$$F^*(X) = B_0 + B_1X + B_2X^2 + \cdots + B_nX^n, \qquad B_n\neq 0,$$

be the exponential matric polynomial associated with A. Since the matrices B_i satisfy the condition in Theorem 3, B_0 is an identity for $A=F^*(E)$. The same condition (Theorem 3) also implies that $B_iB_j=B_jB_i$ and hence that the right side of

$$(A - B_0)^r = (B_1 + B_2 + \cdots + B_n)^r$$

may be evaluated by the multinomial theorem. For $r=n$ and $n+1$, the formulas in Theorem 3 yield:

$$(A - B_0)^n = (n!)B_n \neq 0 \quad \text{and} \quad (A - B_0)^{n+1} = 0.$$

Thus, if a quasi-idempotent matrix A is associated with an exponential matric polynomial of degree n, then there exists an idempotent matrix B_0 which is an identity for A and such that $A-B_0$ is nilpotent of index $n+1$.

On the other hand, let A be a matrix for which there is an idempotent matrix B_0, which is an identity for A, and such that $A-B_0$ is nilpotent of index $n+1$. Consider the matric polynomial $F^*(X)$ given below:

$$(1)\quad B_0 + (A - B_0)X + [(A - B_0)^2/2!]X(X - E) + \cdots + [(A - B_0)^n/n!]X(X - E)\cdots(X - nE + E).$$

Since $A^r=[B_0+(A-B_0)]^r$, it follows from the binomial theorem and the nilpotency of $A-B_0$ that $F^*(rE)=A^r$ for every positive integer r. Hence, A is quasi-idempotent and $F^*(X)$ is the unique exponential matric polynomial associated with A. The unicity of $F^*(X)$ assures that a quasi-idempotent matrix A has associated with it a unique matrix B_0 which is an identity for A and such that $A-B_0$ is nilpotent.

THEOREM 4. *A matrix A is quasi-idempotent if and only if there is an idempotent matrix B_0 which is an identity for A and such that $A-B_0$ is nilpotent. The associated exponential matric polynomial $F^*(X)$ is computed by* (1) *and its degree is one less than the index of nilpotency of $A-B_0$.*

Of course, the associated exponential polynomial matrix $F(x)$ is the image of the associated exponential matric polynomial $F^*(X)$ in the natural isomorphism. For a given matrix A, it might be difficult to establish the existence or nonexistence of an idempotent matrix B_0 satisfying the conditions of Theorem 4. This may be remedied as follows:

If the matrix B_0 is an identity for the matrix A, consider the identity

$$(*) \qquad (B_0-A)^k-(E-A)^k=B_0-E, \qquad \text{for any positive integer } k.$$

If the matrix A is quasi-independent and B_0 is the associated idempotent matrix such that $(A-B_0)^{n+1}=0$, then it follows from (*) that $B_0=E-(E-A)^{n+1}$. If both sides of this equation are multiplied on the left by A, the resulting equation is equivalent to $A(E-A)^{n+1}=0$. Hence, if the matrix A is quasi-idempotent and is associated with an exponential matric polynomial of degree n, then $A(E-A)^{n+1}=0$.

Conversely, let A be a matrix such that $A(E-A)^k=0$ for some positive integer k and define B_0 by the equation $B_0=E-(E-A)^k$. If a matrix C has the property that it can be written in the form AD, where D commutes with A, then it is easily verified that $CB_0=B_0C=C$. Since the matrix B_0 has the property required of C, then $B_0B_0=B_0$ and B_0 is idempotent. Since A also has the property required of C, then B_0 is an identity for A. Finally, it follows from the identity (*) and $B_0=E-(E-A)^k$ that $(B_0-A)^k=0$ and hence, by Theorem 4, that A is quasi-idempotent.

THEOREM 5. *A matrix A is quasi-idempotent if and only if $A(A-E)^k=0$ for some positive integer k. If n is a positive integer such that $A(A-E)^n\neq 0$ and $A(E-A)^{n+1}=0$, then $B_0=E-(E-A)^{n+1}$ is the constant term in the exponential matric polynomial $F^*(X)$ associated with A and $F^*(X)$ may be computed by* (1).

The integer n of Theorem 5 will be called the *index* of the quasi-idempotent matrix A. It is the degree of the associated matric polynomial and is one less than the index of nilpotency of $A-B_0$.

3. Computation of the exponential matric polynomial associated with a quasi-idempotent matrix. If A is quasi-idempotent of index n, the constant term

in the associated exponential matric polynomial is given by the formula $B_0 = E-(E-A)^{n+1}$. The coefficient of X in $F^*(X)$ is determined from (1) to be:

$$(2)\quad B_1 = (A - B_0) - (A - B_0)^2/2 + (A - B_0)^3/3 + \cdots + (-1)^{n-1}(A - B_0)^n/n.$$

It follows directly from Theorem 3 that $B_i = B_1^i/i!$ for $i>0$. These give explicit formulas for computing the associated matric polynomial $F^*(X)$ and also lead naturally to the following notions. Since $A = F^*(E)$,

$$(3)\quad A = B_0 + B_1 + B_1^2/2! + \cdots + B_1^n/n!.$$

Except for the appearance of B_0, which is an identity for A, in place of the universal identity matrix E and for the fact that these are finite sums, the formulas of (2) and (3) are the same as the usual series for the logarithmic and exponential functions of matrices. This formal resemblance suggests investigating the following definitions in light of the present ideas:

DEFINITION 1. *If A is quasi-idempotent, then* $\ln A$ *is the coefficient of X in the unique exponential matric polynomial associated with A.*

DEFINITION 2. *If N is a nilpotent matrix of index $n+1$ and B_0 is an identity for N, then* $\exp_{B_0} N = B_0+N+N^2/2!+\cdots+N^n/n!$.

In this language, (2) and (3) now take the form:

THEOREM 6. *If A is a quasi-idempotent matrix of index n and B_0 is the idempotent matrix associated with A, then*

$$\ln A = (A - B_0) - (A - B_0)^2/2 + \cdots + (-1)^{n-1}(A - B_0)^n/n$$

and $A = \exp_{B_0}(\ln A)$.

Two examples will be given to show how the ideas in sections 1 and 2 are sufficient to prove that the functions defined have appropriate properties.

If B_0 is an identity for a nilpotent matrix N of index $n+1$, then the matric polynomial $G^*(X) = B_0+NX+(N^2/2!)X^2+\cdots+(N^n/n!)X^n$ satisfies the characteristic conditions of Theorem 3 and is hence an exponential matric polynomial. The exponential matric polynomial $G^*(X)$ is associated with the quasi-idempotent matrix $G^*(E) = \exp_{B_0} N$, and the matrix $\ln(\exp_{B_0} N)$ is the coefficient of X in $G^*(X)$ or N.

THEOREM 7. *If B_0 is an identity for a nilpotent matrix N of index $n+1$,* then $\exp_{B_0} N$ *is quasi-idempotent of index m and* $\ln(\exp_{B_0} N) = N$.

If A and C are commuting quasi-idempotent matrices associated with the exponential matric polynomials $F^*(X) = B_0+B_1X+\cdots$ and $G^*(X) = D_0+D_1X+\cdots$, respectively, then the assumption of commutativity implies that any B_i commutes with any D_j. Then for all positive integers r,

$$(AC)^r = A^rC^r = F^*(rE)G^*(rE) = [F^*(X)G^*(X)]_{X=rE}.$$

This means that $F^*(X)G^*(X)$ is the exponential matric polynomial associated with AC and hence that AC is quasi-idempotent. Moreover, B_0D_0 is the constant term in $F^*(X)G^*(X)$ and the coefficient of X is $B_0D_1+D_0B_1$ or $B_0 \ln C+D_0 \ln A$.

THEOREM 8. *If A and C are commuting quasi-idempotent matrices with associated idempotent matrices B_0 and D_0, then AC is a quasi-idempotent matrix associated with B_0D_0 and* $\ln AC=D_0 \ln A+B_0 \ln C$.

There is the immediate corollary:

COROLLARY. *If A and C are commuting quasi-idempotent matrices associated with the same idempotent matrix, then* $\ln AC$ *exists and* $\ln AC=\ln A+\ln C$.

It follows from Theorem 5 that the idempotent matrix associated with a non-singular quasi-idempotent matrix is the identity matrix E. Thus, any pair of commuting non-singular quasi-idempotent matrices satisfy the conclusion in the Corollary.

Concluding remarks. If A is a quasi-idempotent matrix with the associated idempotent matrix B_0, it follows directly from the ideas above that the set of matrices $G=\{\exp_{B_0}(s \ln A) \mid s \text{ is a complex number}\}$ is a multiplicative group containing A and that $\exp_{B_0}[(1/n) \ln A)]$ is an n-th root of A in G.

The proper role of quasi-idempotent matrices in the classical theory may be indicated as follows: If C is a group element in a multiplicative group, then the minimum function of C is of the form:

$$m(z) = \prod_i (z - c_i)^{e_i},\ i = 1, 2, \cdots, s; \qquad \text{with } c_i \neq 0,$$

or of the form $zm(z)$. In either case, C has a decomposition into a direct sum of the form

$$C = C_1 \oplus C_2 \oplus \cdots \oplus C_s,$$

where $z(z-c_i)^{e_i}$ is the minimum function of C_i. Hence, if $A_i=c_i^{-1}C_i$, then

$$C = c_1A_1 \oplus c_2A_2 \oplus \cdots \oplus c_sA_s,$$

where each A_i is quasi-idempotent of index e_i-1 and $A_iA_j=0$ for $i \neq j$. That is, any group element is a linear combination of mutually orthogonal quasi-idempotent matrices. The associated idempotent matrices are the principal idempotent elements (Frobenius covariants) of C.

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APPLICATIONS OF FAÀ DI BRUNO'S FORMULA IN MATHEMATICAL STATISTICS

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1. History of the problem solved by Faà di Bruno. The problem of finding an explicit expression for the p-th derivative of a function of a function is an old one. Let $z=G(y)$ and $y=f(x)$ be two functions such that all the derivatives of $G(y)$ and $f(x)$ up to order p exist. The problem consists then in finding a formula for

$$\frac{d^p z}{dx^p} = \frac{d^p}{dx^p} G[f(x)].$$

The need for such a formula was already stated in old treatises on calculus such as S. F. Lacroix's [11]; however no satisfactory solution was given. O. Schloemilch [16] gave a formula for the case where $f(x)=x^\lambda$ or where $f(x)=e^x$. By specializing his formulae he obtains many of the expressions for successive derivatives of algebraic functions which were known at his time and also some formulae ascribed to Euler and Laplace for the successive derivatives of $1/(e^x \pm a)$.

Reinhold Hoppe realized that the problem could be solved for all functions $f(x)$ for which the higher derivatives of $[f(x)]^k$ (k a positive integer) are explicitly known. In 1845 he published a monograph [7] on this subject and, at the request of the editor of Crelle's Journal, he also wrote a paper [8] indicating the contents of his book.

In 1870 R. Götting [5] considered the case $G(y)=y^k$ and derived a number of formulae for suitably specialized choices of $f(x)$. This induced Hoppe to restate his more general result in a brief note [9]. Hoppe's book was not accessible to the present writer. His information about Hoppe's work is based on the two papers [8], [9] mentioned above.

The first solution of the problem which carried no restrictions as to the functions $G(y)$ and $f(x)$ was given by Cavaliere Francesco Faà di Bruno in 1855. He published two almost identical papers, one in the *Annali di Tortolini* [2], the second in the *Quarterly Journal of Mathematics* [3]. In these papers he gave his formula essentially in the form in which we quote it in the next section. He also wrote the formula in the form of a determinant. About twenty years later Faà di Bruno published a book [4] on binary forms. The formula is given on page 4 while its proof is found in an appendix (pp. 304–305). R. Most [15] also wrote a brief paper on this subject. He quotes [5] and [9] but was apparently not aware of Faà di Bruno's work. He obtained a general formula. His method could also be applied in case G is a function of several variables each of which depends on x. The proof in Faà di Bruno's book is rather complicated. A simple proof by induction was given by L. Koenigsberger [10]. Another simple proof, assuming that both $G(y)$ and $f(x)$ are analytic in a neighborhood of the origin, is due to Fr. Meyer [14] who discussed also the algebraic character

of Faà di Bruno's formula. A good survey of the problem and its history may be found in section 24 of A. Voss' article [17] on calculus in the *Enzyklopädie*.

Faà di Bruno's formula is nowadays almost forgotten and rarely occurs in modern textbooks of Calculus. It may, however, be found in Goursat's *Cours d'Analyse* [6]. The purpose of the present paper is to revive interest in Faà di Bruno's formula and to show that it has worthwhile applications in mathematical statistics. Some of the formulae discussed in the subsequent sections are well known but usually presented without proof. One of the results (Theorem 3) has not yet been published.

2. The formula of Faà di Bruno. We assume now that $z=G(y)$ and $y=f(t)$ have derivatives of all orders up to order p and write for simplicity

$$f_v = \frac{1}{v!}\frac{d^v}{dt^v}f(t), \qquad f = f_0 = f(t). \tag{2.1}$$

Faà di Bruno's formula is then

$$\frac{d^p z}{dt^p} = \frac{d^p}{dt^p}G[f(t)] = \sum \frac{d^k G(y)}{dy^k}\,\frac{p!}{i_1!i_2!\cdots i_s!}\,f_{k_1}^{i_1}f_{k_2}^{i_2}\cdots f_{k_s}^{i_s}, \tag{2.2}$$

where the summation is to be extended over all partitions of p such that

$$\begin{cases} i_1 + i_2 + \cdots + i_s = k \\ i_1k_1 + i_2k_2 + \cdots + i_sk_s = p. \end{cases} \tag{2.3}$$

We do not intend to prove (2.2) here. The reader can easily establish this formula by induction or he may consult the references [10] or [14]. However, we list two particular cases which occur in the statistical applications. We put first $G(y)=\ln y$ so that

$$\frac{d^k G(y)}{dy^k} = (-1)^{k-1}(k-1)!y^{-k}$$

and obtain from (2.2) and (2.3)

$$\frac{d^p}{dt^p}\ln f(t) = \sum (-1)^{k-1}\frac{p!(k-1)!}{i_1!i_2!\cdots i_s!}(f_{k_1}/f)^{i_1}(f_{k_2}/f)^{i_2}\cdots(f_{k_s}/f)^{i_s}. \tag{2.4}$$

We next put $G(y)=e^y$ and write $\phi(t)$ instead of $f(t)$; then

$$\frac{d^p}{dt^p}e^{\phi(t)} = \sum e^{\phi(t)}\,\frac{p!}{i_1!i_2!\cdots i_s!}\,\phi_{k_1}^{i_1}\phi_{k_2}^{i_2}\cdots\phi_{k_s}^{i_s}. \tag{2.5}$$

In (2.4) and (2.5) the summation is to be taken over all partitions of p for which (2.3) holds.

3. Cumulants and moments. In the following sections we denote by $F(x)$ a distribution function, that is, a never decreasing function which is continuous to

the right and for which $F(-\infty)=0$, $F(+\infty)=1$. The Fourier transform

$$f(t) = \int_{-\infty}^{+\infty} e^{itx} dF(x)$$

exists for any distribution function $F(x)$ and is called its characteristic function. It is easy to see that $f(t)$ is a continuous function of t and that $f(0)=1$. It follows then that a characteristic function does not vanish in some neighborhood of $t=0$. We may, therefore, consider its logarithm

$$\phi(t) = \ln f(t)$$

at least in this neighborhood. This function is sometimes called the cumulant generating function of the distribution $F(x)$. The number

$$\alpha_k = \int_{-\infty}^{+\infty} x^k dF(x)$$

is called the moment of order k of the distribution $F(x)$. It is known (see [1] chapter 10) that the existence of the moment of order k of a distribution implies that its characteristic function may be differentiated k times and that

$$\alpha_v = \frac{1}{i^v} f^{(v)}(0) \qquad v = 0, 1, 2, \cdots k. \tag{3.1}$$

If the moment of order k exists, then the cumulant generating function $\phi(t)$ is also k times differentiable. The quantities

$$\kappa_v = \frac{1}{i^v} \phi^{(v)}(0) \qquad v = 1, 2, \cdots k \tag{3.2}$$

are called the cumulants (sometimes semi-invariants) of the distribution $F(x)$.

As a first application of Faà di Bruno's formula, we state and prove the following proposition.

THEOREM 1. *Let $F(x)$ be a distribution function and assume that its moment of order m exists. The moments and cumulants of $F(x)$ are then connected by the relations*

$$\kappa_p = \sum (-1)^{k-1} \frac{p!(k-1)!}{i_1!(k_1!)^{i_1} \cdots i_s!(k_s!)^{i_s}} \alpha_{k_1}^{i_1} \cdots \alpha_k^{i_s} \tag{3.3}$$

and

$$\alpha_p = \sum \frac{p!}{i_1!(k_1!)^{i_1} \cdots i_s!(k_s!)^{i_s}} \kappa_{k_1}^{i_1} \cdots \kappa_{k_s}^{i_s} \tag{3.4}$$

for $p=1, 2, \cdots, m$. The summations are extended over all partitions of p satisfying (2.3).

The relation (3.3) follows immediately from (2.4), (3.1) and (3.2). Relation (3.4) is obtained from (2.5) if we let $\phi(t)$ be the cumulant generating function of $F(x)$. Then $f(t)=e^{\phi(t)}$ and we derive (3.4) by setting $t=0$ and considering again (3.1) and (3.2).

4. The k-statistics. The next application of Faà di Bruno's formula deals with certain sampling characteristics of populations. We consider a random variable and suppose that we have taken n independent observations $x_1, x_2, \cdots, x_n$ forming a sample of n. The $x_1, x_2, \cdots, x_n$ are then by assumption n independently and identically distributed random variables; we denote their common distribution function by $F(x)$ and call it the distribution function of the population under consideration. Populations are frequently studied by means of functions of a sample. It is customary to call a function of the observations a statistic.

We define the k-statistic of order p as a symmetric polynomial statistic which has the property that its expectation equals the p-th cumulant κ_p for any population distribution possessing the necessary moments. We denote the k-statistics of order p by $k_p=k_p(x_1, x_2, \cdots, x_n)$ and will use in the following the symbol $\mathbf{E}(\cdots)$ for the expectation.

This definition must be justified by demonstrating that it is possible to determine uniquely for every positive integer p a k-statistic k_p.

As a first step we show by means of an indirect proof that for each p at most one statistic can satisfy the requirements of the definition. Let us, therefore, assume tentatively that two statistics k_p and k_p^* satisfy the conditions of the definition. Then $k_p-k_p^*$ is a symmetric polynomial statistic such that $\mathbf{E}(k_p-k_p^*)=0$. If k_p and k_p^* are not identical, then this is a relation between the moments which is valid for any population distribution function. However, such a relation can not hold for an arbitrary population distribution. Thus the assumption that k_p and k_p^* are two different k-statistics of the same order leads to a contradiction.

Before giving an explicit expression for k_p we introduce some notations, which are used in combinatorial analysis for work with symmetric polynomials.

We write $(k_1^{i_1}k_2^{i_2}\cdots k_s^{i_s})$ for the symmetric polynomial of the n variables $x_1, x_2, \cdots, x_n$ whose terms consist of the product of i_1 powers of degree k_1, i_2 powers of degree k_2, $\cdots$, i_s powers of degree k_s, that is

$$(4.1)\qquad (k_1^{i_1}k_2^{i_2}\cdots k_s^{i_s})=\sum x_1^{k_1}\cdots x_{i_1}^{k_1}x_{i_1+1}^{k_2}\cdots x_{i_1+i_2}^{k_2}\cdots x_k^{k_s},$$

where $i_1+i_2+\cdots+i_s=k$ is the number of factors in each term and where $i_1k_1+i_2k_2+\cdots+i_sk_s=p$ is the degree of the symmetric polynomial. Clearly there are

$$\frac{n(n-1)\cdots(n-k+1)}{i_1!i_2!\cdots i_s!}\text{ terms in the sum }(k_1^{i_1}k_2^{i_2}\cdots k_s^{i_s}),$$

and we have

$$\mathbf{E}\{(k_1^{i_1}k_2^{i_2}\cdots k_s^{i_s})\} = \frac{n(n-1)\cdots(n-k+1)}{i_1!i_2!\cdots i_s!}\alpha_{k_1}^{i_1}\cdots\alpha_{k_s}^{i_s}. \tag{4.2}$$

If we introduce

$$[k_1^{i_1}k_2^{i_2}\cdots k_s^{i_s}] = i_1!i_2!\cdots i_s!(k_1^{i_1}k_2^{i_2}\cdots k_s^{i_s}) \tag{4.3}$$

then we have

$$\mathbf{E}\{[k_1^{i_1}k_2^{i_2}\cdots k_s^{i_s}]\} = n(n-1)\cdots(n-k+1)\alpha_{k_1}^{i_1}\cdots\alpha_{k_s}^{i_s}. \tag{4.4}$$

From (4.4) and (2.4) we obtain easily an explicit expression for the k statistic of order p

$$k_p = \sum(-1)^{k-1}\frac{p!(k-1)!\,[k_1^{i_1}k_2^{i_2}\cdots k_s^{i_s}]}{i_1!(k_1!)^{i_1}\cdots i_s!(k_s!)^{i_s}n(n-1)\cdots(n-k+1)}. \tag{4.5}$$

The summation is to be extended over all partitions of p satisfying (2.3).

It would have been possible to define the statistics k_p by (4.5); such a definition would depend only on the observations and would therefore be valid also for population distributions which do not have moments of order p. However, this generalization of the concept of k-statistics would be of little value since these statistics are interesting mainly on account of the relation

$$\mathbf{E}k_p = \kappa_p \tag{4.6}$$

which we used in our original definition and which has meaning only if the p-th moment of $F(x)$ exists.

We show now an interesting property of the k-statistics which we shall use in the next section.

THEOREM 2. *The k-statistics are invariant under translations, that is for any real a we have*

$$k_p(x_1+a,\ x_2+a,\cdots,\ x_n+a) = k_p(x_1,\ x_2,\cdots x_n). \tag{4.7}$$

To prove Theorem 2 we introduce the random variable $y=x+a$ and remark that for all orders exceeding one, x and y have the same cumulants. We consider now the sample $y_1\cdots y_n$ where $y_j=x_j+a$. This has a population distribution function $G(y)=F(y-a)$. Let $p>1$, then

$$\mathbf{E}\{k_p(x_1+a,\cdots,\ x_n+a)\} = \mathbf{E}\{k_p(y_1,\cdots,\ y_n)\} = \kappa_p.$$

Equation (4.7) follows then immediately from the last relation and from the uniqueness of the k-statistics.

5. A characterization of the normal distribution. In this section we characterize a population with a normal population distribution function by a property

of the k-statistics.

The distribution function

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x} e^{-(y-a_1)^2/2\sigma^2}dy$$

is called the normal distribution. Its characteristic function is given by $e^{it a_1}e^{-\sigma^2 t^2/2}$. A population with a normal population distribution is called a normal population and can be characterized in the following manner:

THEOREM 3. *Let $x_1, x_2, \cdots, x_n$ be n independent observations taken from a population with distribution function $F(x)$ and denote by p an integer greater than one. Assume that the p-th moment of $F(x)$ exists. The population is normal if, and only if, the k-statistic of order p is independent of the sample mean.*

We first prove the sufficiency of the condition. The independence of k_p and the sample mean $(x_1+x_2+\cdots+x_n)/n$ means that the two statistics $Y=k_p$ and $L=x_1+x_2+\cdots+x_n$ are independent. This fact may be expressed in terms of the characteristic functions of these statistics. The characteristic function of Y and L can be written as $\mathbf{E}\{e^{itL+iuY}\}$; it is well known that the independence of L and Y may be expressed by means of the relation

$$\mathbf{E}\{e^{itL+iuY}\} = \mathbf{E}\{e^{itL}\}\mathbf{E}\{e^{iuY}\}. \tag{5.1}$$

The right side of (5.1) is the product of the characteristic functions of the random variables L and Y respectively. We remark that the relation (5.1) may be differentiated with respect to u because the p-th moment of $F(x)$ exists. We differentiate (5.1) with respect to u and put afterwards $u=0$ and obtain

$$\mathbf{E}\{Ye^{itL}\} = \mathbf{E}\{Y\}\mathbf{E}\{e^{itL}\} = \mathbf{E}\{Y\}[f(t)]^n. \tag{5.2}$$

We must next compute $\mathbf{E}\{Ye^{itL}\}$. Substituting for $Y=k_p$ the expression (4.5), we obtain

$$\mathbf{E}\{Ye^{itL}\} = \mathbf{E}\left\{\sum (-1)^{k-1}\frac{p!(k-1)!\,[k_1^{i_1}\cdots k_s^{i_s}]e^{itL}}{i_1!(k_1!)^{i_1}\cdots i_s!(k_s!)^{i_s}n(n-1)\cdots(n-k+1)}\right\}.$$

The sum is to be extended over all partitions of p satisfying (2.3). We interchange the operations of summation and expectation and obtain

$$\mathbf{E}\{Ye^{itL}\} = \sum (-1)^{k-1}\frac{p!(k-1)!\,\mathbf{E}\{[k_1^{i_1}\cdots k_s^{i_s}]e^{itL}\}}{i_1!(k_1!)^{i_1}\cdots i_s!(k_s!)^{i_s}n(n-1)\cdots(n-k+1)}. \tag{5.3}$$

The existence of the p-th moment of $F(x)$ implies that the characteristic function $f(t)$ may be differentiated p times and that

$$\frac{d^r f(t)}{dt^r} = i^r \int_{-\infty}^{\infty} x^r e^{itx} dF(x), \qquad (r = 0, 1, \cdots, p).$$

From this it is easily seen that the typical term in the sum $[k_1^{i_1} \cdots k_s^{i_s}]$ has the same expectation

$$i^{-p}[f(t)]^{n-k}[f^{k_1}(t)]^{i_1} \cdots [f^{k_s}(t)]^{i_s}.$$

Since the $n(n-1) \cdots (n-k+1)$ terms of this sum have the same expectation, we obtain

$$\mathbf{E}\{[k_1^{i_1} \cdots k_s^{i_s}]e^{itL}\} = n(n-1) \cdots (n-k+1) i^{-p}[f(t)]^{n-k}[f^{k_1}(t)]^{i_1} \cdots [f^{k_s}(t)]^{i_s}.$$

We substitute this into (5.3) and use again the notation introduced by (2.1) and obtain

$$\mathbf{E}\{Ye^{itL}\} = i^{-p}[f(t)]^n \sum (-1)^{k-1} \frac{p!(k-1)!}{i_1! \cdots i_s!} [f_{k_1}/f]^{i_1} \cdots [f_{k_s}/f]^{i_s} \tag{5.4}$$

where the summation is to be extended over all partitions of p for which (3.2) holds.

From (5.2), (5.4) and Faà di Bruno's formula (2.4) we obtain easily the differential equation

$$\frac{d^p}{dt^p} \ln f(t) = i^p \kappa_p. \tag{5.5}$$

To obtain this differential equation we had to divide by $[f(t)]^n$. It is, therefore, valid in any region in which $f(t)$ is different from zero. The fact that $f(0) = 1$ and that $f(t)$ is continuous assures the existence of a neighborhood of the origin in which this condition is satisfied. We restrict our considerations first to such a neighborhood of $t = 0$. Using the initial conditions suggested by (3.2), this equation can be integrated and we obtain

$$f(t) = \exp\left\{\sum_{j=1}^{p} \kappa_j (it)^j / j!\right\}. \tag{5.6}$$

This solution is valid in any neighborhood of the origin in which $f(t)$ is different from zero. We show by an indirect proof that $f(t)$ has no real zero so that (5.6) is valid for all t.

Let us, therefore, assume that $f(t)$ has a zero and that t_0 is the zero closest to the origin. Then (5.6) holds for $t < t_0$ and we conclude from this and from the continuity of $f(t)$ that

$$\lim_{t \to t_0} f(t) = \exp\left\{\sum_{j=1}^{p} \kappa_j (it_0)^j / j!\right\}.$$

This, however, contradicts the assumption $f(t_0)=0$ so that the validity of (5.6) for all real t is proven.

All the solutions of the differential equations (5.5) are given by (5.6); however, not all these solutions are characteristic functions. It is still necessary to select from the solutions (5.6) those functions which are characteristic functions. This can be accomplished by means of the following result due to Marcinkiewicz [13].

THEOREM *of Marcinkiewicz. No function of the form* $\exp[a_0+a_1z+\cdots+a_rz^r]$ *with* $r>2$ *can be a characteristic function.*

The degree of the polynomial in (5.6) can, therefore, not exceed two, *i.e.*, $\kappa_j=0$ for $j>2$ and

$$f(t) = e^{i\kappa_1 t-\kappa_2 t^2/2}. \tag{5.7}$$

This is the characteristic function of the normal distribution and we conclude from the uniqueness theorem (see [1]) that the sufficiency of the condition of Theorem 2 is established.

The necessity of the condition follows from Theorem 2 and the well known fact that in a normal population every translation invariant statistic is independent of the mean.

Finally we should like to note that a remark in a paper by R. G. Laha [12] indicates that D. Basu and R. G. Laha obtained independently the result of Theorem 3 and that it will be published in *Sankhya*. The author of the present paper learned from Mr. Basu that their proof of Theorem 3 is in many respects similar to the one given above, but that it makes no use of Faà di Bruno's formula.

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A NOTE ON THE EVEN PERFECT NUMBERS

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1. The harmonic mean. Following Ore,* we define $H(n)$ to be the harmonic mean of the divisors of n, so that

$$\frac{1}{H(n)} = \frac{1}{\tau(n)} \sum_{d|n} \frac{1}{d} = \frac{1}{n\tau(n)} \sum_{d'|n} d' = \frac{\sigma(n)}{n\tau(n)},$$

where $\tau(n)$ is the number of divisors of n, $\sigma(n)$ is the sum of the divisors of n, and $dd' = n$. Thus we have

$$H(n) = \frac{n\tau(n)}{\sigma(n)}, \tag{1}$$

from which it is clear that $H(n)$ is multiplicative.

Computation shows that when p is a prime

$$H(p^a) = \frac{p^a(a+1)}{1 + p + \cdots + p^a}, \qquad a \geqq 1. \tag{2}$$

When p is an odd prime and $a \geqq 2$, we may use (2) and an induction on a to show $H(p^a) > 2$. Similarly, when $p = 2$ and $a \geqq 3$, we have $H(2^a) > 2$. When p and q are distinct primes, we have

$$H(pq) = \frac{4pq}{(p+1)(q+1)} \geqq 2,$$

for this inequality reduces to $(p-1)(q-1) \geqq 2$; we note that the equality holds only when $p = 2$, $q = 3$. When p is an odd prime,

* On the averages of the divisors of a number, O. Ore, this MONTHLY, vol. 55, 1948, pp. 614–19.

$$H(4p) = \frac{24p}{7(p+1)} > 2.$$

Combining these results with the fact that $H(n)$ is multiplicative, we find that $H(n)>1$ when $n>1$, and that $H(n)>2$, except when $n=p$, a prime, and when $n=1$, 4 and 6. In particular, we need the result that $H(n)>2$ for every odd composite number.

2. Even perfect numbers. After Euclid and Euler we know that the even perfect numbers (defined by requiring $\sigma(n)=2n$) are exactly those numbers of the form $n=2^{q-1}(2^q-1)$ where $p=2^q-1$ is prime. Ore has already noted that for such an even perfect number $H(n)=n(2q)/2n=q$.

The object of this note is to show the following converse.

THEOREM: *If a given integer n is even and has the form*

$$n = 2^{H(n)-1}(2^{H(n)} - 1), \tag{3}$$

then n must be a perfect number.

Proof: In view of the Euclid-Euler theorem it will suffice to prove in (3) that $P=2^{H(n)}-1$ is prime. Since n is assumed even, we know $H(n)>1$. From (3) we compute

$$H(n) = \frac{2^{H(n)-1}H(n)}{2^{H(n)} - 1} H(P) > \frac{1}{2} H(n)H(P).$$

Hence $H(P)<2$ and by the remark at the end of the first section, we know that P, although odd and greater than 1, is not composite.

MATHEMATICAL NOTES

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SUMS OF POWERS OF NUMBERS HAVING A GIVEN PERIOD MODULO m

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Raymond Moller [3, Th. 1] showed that if the numbers a_i are the numbers less than a prime p having period e' (mod p), then the sum of the nth powers

$$\sum a_i^n \equiv \frac{\phi(e')}{\phi(e'/(n, e'))} \mu(e'/(n, e')) \pmod{p},$$

where ϕ is the Euler ϕ-function and μ is the Möbius function. We proceed to

establish similar theorems for composite moduli.

THEOREM 1. *The sum of the positive integers less than p^s having period $e \equiv p^r e' \pmod{p^s}$, where $0 \leqq r < s$, $e' \mid (p-1)$, $s \geqq 1$, and p is an odd prime is congruent to $\phi(p^r)\mu(e') \pmod{p^s}$.*

*Proof.** Let S_e denote the sum (mod p^s) of those residues (mod p^s) whose period is a divisor of e. Let $\sum_e$ denote the sum of the residues (mod p^s) whose period equals e. Then

$$S_e = \sum_{d \mid e} \sum_d. \tag{1}$$

By the Möbius inversion formula [2]

$$\sum_e = \sum_{d \mid e} \mu(e/d) S_d. \tag{2}$$

Let a be a primitive root (mod p^s). If $(d, p-1) \neq 1$,

$$S_d \equiv \sum_{k=0}^{d-1} a^{k\phi(p^s)/d} \equiv \frac{1 - a^{\phi(p^s)}}{1 - a^{\phi(p^s)/d}} \equiv 0 \pmod{p^s}. \tag{3}$$

If $d = p^t$, then

$$S_d = \sum_{k=0}^{p^t-1} a^{k\phi(p^{s-t})} \equiv \sum_{k=0}^{p^t-1} (1 + m_k p^{s-t}) \equiv p^t + p^{s-t} \sum_{k=0}^{p^t-1} m_k \pmod{p^s}$$

and $0 \leqq m_k < p^t$. But $m_j \neq m_k$, $j \neq k$, since assuming the contrary $a^{j\phi(p^{s-t})} \equiv a^{k\phi(p^{s-t})} \pmod{p^s}$ implies $a^{(j-k)\phi(p^{s-t})} \equiv 1 \pmod{p^s}$ contrary to the hypothesis that a is primitive (mod p^s). Thus the m_k are the integers 0 to $p^t - 1$ in some order. Thus

$$S_d \equiv p^t + p^{s-t} p^t \frac{p^t - 1}{2} \equiv p^t \pmod{p^s}. \tag{4}$$

Substituting (3) and (4) in (2) and setting $e = e' p^r$ we have

$$\begin{aligned} \sum_e &= \sum_{d \mid e} \mu(e/d) S_d = \sum_{t \leqq r} \mu(e' p^{r-t}) S_{p^t} \\ &= \mu(e') p^r + \mu(e' p) p^{r-1} \equiv \mu(e') \phi(p^r). \end{aligned} \tag{5}$$

THEOREM 2. *The sum of the nth powers of the positive integers less than p^s having period $e = p^r e' \pmod{p^s}$, where $0 \leqq r < s$, $e' \mid (p-1)$, $s \geqq 1$, and p is an odd prime, is congruent to*

$$\frac{\phi(e')}{\phi(e'/(n, e'))} \phi(p^r) \mu(e'/(n, e')) \pmod{p^s}.$$

Proof. The proof of Moller's Theorem 1 [3] holds as well for this case.

LEMMA 1. *For $s > 2$ and $0 < r < s-1$, the integers a_i belonging to $2^r \pmod{2^s}$,*

* This proof was suggested by a referee.

in addition to 2^s-1 for $r=1$, are $a_i=\pm 1+2^{s-r}\sigma_i$, where the σ_i are the positive odd numbers less than 2^r.

Proof. For $r=1$, $(\pm 1+2^{s-1}\sigma_i)^{2^0}\not\equiv 1 \pmod{2^s}$ and

$$(\pm 1 + 2^{s-1}\sigma_i)^{2^1} = 1 \pm 2^s\sigma_i + 2^{2s-2}\sigma_i^2 \equiv 1 \pmod{2^s},$$

so that a basis is established.

State as induction hypothesis that the $\pm 1+2^{s-(r-1)}\sigma_i$ belong to $2^{r-1} \pmod{2^s}$. Now

$$(\pm 1 + 2^{s-r}\sigma_i)^2 = 1 \pm 2^{s-r+1}\sigma_i + 2^{2s-2r}\sigma_i^2 = 1 \pm 2^{s-(r-1)}(\sigma_i \pm 2^{s-r-1}\sigma_i^2)$$

$$\equiv 1 + 2^{s-(r-1)}\sigma_{i'}$$

or

$$1 - 2^{s-(r-1)}(2^{r-1} - \sigma_{j'}) \equiv 1 + 2^{s-(r-1)}\sigma_{j'}.$$

By the induction hypothesis, $1+2^{s-(r-1)}\sigma_{j'}$ belongs to $2^{r-1} \pmod{2^s}$. Then $(\pm 1+2^{s-r}\sigma_i)^{2^r}\equiv(1+2^{s-(r-1)}\sigma_{j'})^{2^{r-1}}\equiv 1 \pmod{2^s}$ and $(\pm 1+2^{s-r}\sigma_i)^{2^t}\equiv 1$ for $t<r$ would imply that $(1+2^{s-(r-1)}\sigma_{j'})^{2^{t-1}}\equiv 1$ for $t-1<r-1$. Therefore $\pm 1+2^{s-r}\sigma_i$ belongs to $2^r \pmod{2^s}$.

Since we have found 2^r integers belonging to $2^r \pmod{2^s}$ for each positive r less than $s-1$ and since, in addition, 1 belongs to 2^0 and 2^s-1 to $2^1 \pmod{2^s}$, we have accounted for all the 2^{s-1} odd positive integers less than 2^s.

THEOREM 3. *For $s>2$, $1<r<s-1$, the sum of the nth powers of the positive integers less than 2^s having period 2^r is congruent to zero for n odd and to 2^r for n even. For $r=1$ the sum is congruent to -1 for n odd and $s>1$, to 1 for n even and $s=2$, and to 3 for n even and $s>2$. For $s>0$, the sum of the nth powers of 1, which belongs to 1, is congruent to 1.*

Proof. Using the representation obtained in Lemma 1, we have for odd n the pairing

$$(\pm 1 + 2^{s-r}\sigma_i)^n \equiv -(\mp 1 + 2^{s-r}(2^r - \sigma_i))^n,$$

giving $\sum a_i^n\equiv 0$ for $s>2$ and $1<r<s-1$. For $s>2$ and $r=1$ we have in addition $a_3\equiv -1$, so that $\sum a_i^n\equiv -1$.

For n even, say $n=2y$, we have

$$\sum_i a_i^{2y} = \sum_i (1 + 2^{s-r}\sigma_i)^{2y} + \sum_i (-1 + 2^{s-r}\sigma_i)^{2y}$$

$$= \sum_i 1 + \sum_i 2y2^{s-r}\sigma_i + \sum_i \sum_{j=2}^{2y} C_j^{2y} 2^{j(s-r)}\sigma_i^j$$

$$+ \sum_i 1 - \sum_i 2y2^{s-r}\sigma_i + \sum_i \sum_{j=2}^{2y} C_j^{2y}(-1)^{2y-j}2^{j(s-r)}\sigma_i^j.$$

The two sums $\sum_i 1$ sum to 2^r. The following two sums cancel. It can be shown by induction on r that $\sum \sigma_i^j$ for $j>1$ is divisible by 2^{r-1}. Thus the term $\pm \sum_i C_j^{2y} 2^{j(s-r)} \sigma_i^j$ for any $j>1$ is divisible by $2^{j(s-r)+r-1} = 2^{s+(j-1)(s-r)-1}$. Then, since $s-r>1$, this term is congruent to zero. Therefore, $\sum a_i^n \equiv 2^r$ for n even, $s>2$, and $1<r<s-1$. Including $a_3 \equiv -1$ for $r=1$, we obtain $\sum a_i^n \equiv 3$.

THEOREM 4. *Let the modulus m be a product of distinct prime powers $m=m_1m_2 \cdots m_k$, where $m_r=p_r^{\alpha_r}$. Let the a_i be the distinct positive integers less than m having period e* (mod m). *Let $e_1, e_2, \cdots, e_k$ be divisors of e such that $e=[e_1, e_2, \cdots, e_k]$. Let the b_{i_r} be the N_r distinct positive integers less than m_r having period e_r* (mod m_r). *(If e_r does not divide $\phi(m_r)$, then there are no b_{i_r} and N_r equals zero.) Then the sum of the nth powers of the distinct positive integers less than m with period e* (mod m) *is a solution* (mod m) *of the k simultaneous congruences*

$$\sum a_i^n \equiv \sum_D N_1N_2 \cdots N_{r-1}N_{r+1} \cdots N_k \sum_{i_r} b_{i_r}^n \pmod{m_r}$$

$$\text{for} \quad r = 1, 2, \cdots, k,$$

the sums $\sum_D$ taken over all decompositions $e_1, e_2, \cdots, e_k$ of e.

Note that $\sum_{i_r} b_{i_r}^n$ is evaluated by Theorems 2 and 3. Whenever N_r is nonzero, that is when $e_r \mid \phi(m_r)$, N_r is known to equal $\phi(e_r)$. (This can be deduced from [1] arts. 85–89.)

Proof. Let the residues of a_i (mod m_r) for $r=1, 2, \cdots, k$ be $b_1(i)$, $b_2(i)$, $\cdots$, $b_k(i)$. We shall show that each a_i is in correspondence with a set of $b(i)$'s whose periods form a decomposition of e such that $e=[e_1, e_2, \cdots, e_k]$.

For a given a_i consider the $b_r(i)$. We shall show that their periods constitute a set $e_1, e_2, \cdots, e_k$ such that $e=[e_1, e_2, \cdots, e_k]$. Since $a_i^e \equiv 1$ (mod m) and therefore $a_i^e \equiv 1$ (mod m_r), each of the periods is a divisor of e. Let L be the least common multiple of the periods. Then $a_j^L \equiv 1$ (mod m_r), so that $a_i^L = 1 + f_r m_r$ for some integer f_r, for each $r=1, 2, \cdots, k$. Then $f_1m_1 = f_2m_2 = \cdots = f_km_k$. But since the m_r are relatively prime, each f_rm_r is divisible by m so that $a_i^L \equiv 1$ (mod m), and $e \mid L$. Thus the periods of the b_r (mod m_r) constitute a set $e_1, e_2, \cdots, e_k$ such that $e=[e_1, e_2, \cdots, e_k]$.

Conversely, every such set of exponents gives rise to an a_i, namely a solution (mod m) of the simultaneous congruences $a_i \equiv b_r$ (mod m_r), where b_r has period e_r (mod m_r), $r=1, 2, \cdots, k$.

Among the $b_r(i)$ for any one r the set of N_r distinct b_{i_r}'s having period e_r (mod m_r) is repeated $N_1N_2 \cdots N_{r-1}N_{r+1} \cdots N_k$ times. Now $\sum a_i^n \equiv \sum b_r(i)^n$ (mod m_r) for each $r=1, 2, \cdots, k$. Dividing the $b_r(i)$'s into like sets of distinct b_{i_r}'s, we have

$$\sum a_i^n \equiv \sum_D N_1N_2 \cdots N_{r-1}N_{r+1} \cdots N_k \sum_{i_r} b_{i_r}^n \pmod{m_r}.$$

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ON THE INFINITUDE OF PRIMES

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In this note we would like to offer an elementary "topological" proof of the infinitude of the prime numbers. We introduce a topology into the space of integers S, by using the arithmetic progressions (from $-\infty$ to $+\infty$) as a basis. It is not difficult to verify that this actually yields a topological space. In fact, under this topology, S may be shown to be normal and hence metrizable. Each arithmetic progression is closed as well as open, since its complement is the union of other arithmetic progressions (having the same difference). As a result, the union of any finite number of arithmetic progressions is closed. Consider now the set $A = \bigcup A_p$, where A_p consists of all multiples of p, and p runs through the set of primes $\geqq 2$. The only numbers not belonging to A are -1 and 1, and since the set $\{-1, 1\}$ is clearly not an open set, A cannot be closed. Hence A is not a finite union of closed sets which proves that there are an infinity of primes.

A STATISTICAL DERIVATION OF A PAIR OF TRIGONOMETRIC INEQUALITIES

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The following inequalities and a particular generalization of them can be obtained by comparing the variances of a pair of minimum variance estimators with the corresponding variances of certain less efficient estimators:† Given $\theta = (\theta_1, \theta_2, \cdots, \theta_n)$ such that $0 \leqq \theta_i < \pi$ and $\theta_j \neq \theta_k$ for $j \neq k$. Then

$$\frac{\sum_{i=1}^{n} \cos^2 \theta_i}{\sum\sum_{j>i} \sin^2 (\theta_i - \theta_j)} \leqq \binom{n}{2}^{-2} \sum_{i=1}^{n} \left\{ \frac{\cos \theta_1}{\sin (\theta_i - \theta_1)} + \cdots + \frac{\cos \theta_{i-1}}{\sin (\theta_i - \theta_{i-1})} + \frac{\cos \theta_{i+1}}{\sin (\theta_i - \theta_{i+1})} + \cdots + \frac{\cos \theta_n}{\sin (\theta_i - \theta_n)} \right\}^2 \tag{1}$$

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† So far as the author has been able to determine, the inequalities stated in this paper appear to be new. The statistics involved can be considered as respective pairs of estimators of the coordinates of a fixed point in a plane, based on line-of-sight observations. The minimum variance estimators (under the conditions stated) are the coordinates of the point from which the sum of squares of the distances to each line-of-sight is a minimum. The less efficient estimators correspond to the estimate (of the location of the fixed point) obtained by averaging arithmetically the vectors determined by the intersections of all pairs of lines of sight.

and

$$\frac{\sum_{i=1}^{n} \sin^2 \theta_i}{\sum\sum_{j>i} \sin^2 (\theta_i - \theta_j)} \leqq \binom{n}{2}^{-2} \sum_{i=1}^{n} \left\{ \frac{\sin \theta_1}{\sin (\theta_i - \theta_1)} + \cdots + \frac{\sin \theta_{i-1}}{\sin (\theta_i - \theta_{i-1})} \right.$$

(2)

$$\left. + \frac{\sin \theta_{i+1}}{\sin (\theta_i - \theta_{i+1})} + \cdots + \frac{\sin \theta_n}{\sin (\theta_i - \theta_n)} \right\}^2 .$$

Proof: Let $\epsilon_i = z_i - \xi \sin \theta_i - \eta \cos \theta_i$, for $i = 1, 2, \cdots, n$, represent n random independent deviates, with $E(\epsilon_i) = 0$, $E(\epsilon_i^2) = 1$, random components z_i, known parameters θ_i subject to the restrictions stated above, and with unknown parameters ξ and η.

If we let $(\hat{\xi}, \hat{\eta})$ represent the least squares estimate of (ξ, η), *i.e.*, the estimate of (ξ, η) obtained by minimizing $\sum (z_i - \hat{\xi} \sin \theta_i - \hat{\eta} \cos \theta_i)^2$ (*cf.* [1]), then $\hat{\xi}$ and $\hat{\eta}$ are unbiased linear estimates of ξ and η respectively with variances

$$\sigma^2(\hat{\xi}) = \frac{\sum_{i=1}^{n} \cos^2 \theta_i}{\sum_{1}^{n} \sin^2 \theta_i \sum_{1}^{n} \cos^2 \theta_i - \left(\sum_{1}^{n} \sin \theta_i \cos \theta_i \right)^2} = \frac{\sum_{1}^{n} \cos^2 \theta_i}{\sum\sum_{j>i} \sin^2 (\theta_i - \theta_j)} \tag{3}$$

and

$$\sigma^2(\hat{\eta}) = \frac{\sum_{i=1}^{n} \sin^2 \theta_i}{\sum\sum_{j>i} \sin^2 (\theta_i - \theta_j)} . \tag{4}$$

Further, in accordance with the generalized Markoff theorem on least squares (see [2]), $\hat{\xi}$ and $\hat{\eta}$ are minimum variance unbiased estimates of ξ and η in the class of unbiased linear estimates.

Thus any other pair of unbiased linear estimators (x, y) of (ξ, η) will in general have variance greater than (3) and (4), respectively.

Substituting $z_k = \epsilon_k + \xi \sin \theta_k + \eta \cos \theta_k$ for $k = i, j$ in

$$x_{ij} = \frac{z_i \cos \theta_j - z_j \cos \theta_i}{\sin (\theta_i - \theta_j)} \tag{5}$$

shows that, for any i and j, with $i \neq j$, x_{ij} is an unbiased linear estimate of ξ. Therefore the linear function

$$\bar{x} = \binom{n}{2}^{-1} \sum\sum_{j>i} x_{ij} \tag{6}$$

is also an unbiased estimate of ξ. It follows from (5) and (6) by inspection that $\bar{x} = \sum_{i=1}^{n} \delta_i z_i$, where

$$\delta_i = \binom{n}{2}^{-1} \sum_{j \neq i}^{n} \frac{\cos \theta_j}{\sin (\theta_i - \theta_j)}.$$

Hence

$$\sigma^2(\bar{x}) = \sum_{i=1}^{n} \delta_i^2 \sigma^2(z_i) = \sum_{i=1}^{n} \delta_i^2. \tag{7}$$

Since (3), the variance of $\hat{\xi}$, is identical to the left side of (1), and (7) is identical to the right side, the inequality (1) is established.

Analogously, starting with

$$y_{ij} = \frac{z_j \sin \theta_i - z_i \sin \theta_j}{\sin (\theta_i - \theta_j)}$$

and

$$\bar{y} = \binom{n}{2}^{-1} \sum \sum_{j>i} y_{ij},$$

we arrive at the inequality (2).

To obtain a generalization of (1) consider the set of those estimators of ξ, each of which is obtained by averaging arithmetically some combination of the x_{ij}'s as defined by (5). Any such average is an unbiased linear estimate of ξ, and is in general distinct from $\hat{\xi}$. Next corresponding to *each such estimate* define the characteristic function $\alpha_{ij} (= \alpha_{ji})$ as follows:

$\alpha_{ij} = 1$ if x_{ij} is included in the averaging;

$\alpha_{ij} = 0$ otherwise.

The arithmetic average corresponding to a given combination of x_{ij} can then be written as

$$x(\alpha) = \left\{ \binom{n}{2} - r \right\}^{-1} \sum_{j>i} x_{ij} \alpha_{ij}$$

where $\binom{n}{2} - r$ is the number of α_{ij} which do not vanish. The variance of $x(\alpha)$ is then given by the right side of

$$\frac{\sum \cos^2 \theta_i}{\sum \sum_{j>i} \sin^2 (\theta_i - \theta_j)} \leqq \left[\binom{n}{2} - r \right]^{-2} \sum_{i=1}^{n} \left\{ \frac{\alpha_{i1} \cos \theta_1}{\sin (\theta_i - \theta_1)} \right.$$

$$+ \frac{\alpha_{i2} \cos \theta_2}{\sin (\theta_i - \theta_2)} + \cdots + \frac{\alpha_{i,i-1} \cos \theta_{i-1}}{\sin \theta_i - \theta_{i-2}} \tag{8}$$

$$\left. + \frac{\alpha_{i,i+1} \cos \theta_{i+1}}{\sin (\theta_i - \theta_{i+1})} + \cdots + \frac{\alpha_{in} \cos \theta_n}{\sin (\theta_i - \theta_n)} \right\}^2,$$

and thus (8) holds for all possible combinations of zeros and ones in the α_{ij}, (provided at least one α_{ij} is not zero) with appropriate r.

Analogously we have

$$\frac{\sum \sin^2 \theta_i}{\sum_{j>1} \sum \sin^2 (\theta_i - \theta_j)} \leqq \left[\binom{n}{2} - r\right]^{-2} \sum_{i=1}^{n} \left\{ \frac{\alpha_{i1} \sin \theta_1}{\sin (\theta_i - \theta_1)} + \cdots + \frac{\alpha_{i,i-1} \sin \theta_{i-1}}{\sin (\theta_i - \theta_{i-1})} + \frac{\alpha_{i,i+1} \sin \theta_{i+1}}{\sin (\theta_i - \theta_{i+1})} + \cdots + \frac{\alpha_{in} \sin \theta_n}{\sin (\theta_i - \theta_n)} \right\}^2.$$

References

1. Whittaker and Robinson, The Calculus of Observations, 4th Edition, Chapter IX, London, Blackie & Sons, 1940.

2. J. Neyman and F. N. David, Extension of the Markoff theorem on least squares, Statistical Research Memoirs, II, p. 105.

ON THE ISOSCELES TETRAHEDRON*

VICTOR THÉBAULT, Tennie, Sarthe, France

1. THEOREM. *If the centers I_a, I_b, I_c, I_d of the escribed spheres of the truncs opposite the vertices A, B, C, D of a tetrahedron T are situated on the circumsphere of T, then T is isosceles and conversely.*

Proof. Let a, a', b, b', c, c', be the lengths of the edges BC, DA, CA, DB, AB, DC; S_a, S_b, S_c, S_d, R_a, R_b, R_c, R_d be the areas and the radii of the circumcircles of the faces BCD, CDA, DAB, ABC. With reference to T the barycentric coordinates of I_a, I_b, I_c, I_d are

$$(-S_a, S_b, S_c, S_d), \quad (S_a, -S_b, S_c, S_d), \quad (S_a, S_b, -S_c, S_d), \quad (S_a, S_b, S_c, -S_d).$$

The equation of the circumsphere of T in barycentric coordinates (x, y, z, t) is

$$c^2xy + b^2xz + a'^2xt + a^2yz + b'^2yt + c'^2zt = 0. \tag{1}$$

Hence the conditions that I_a, I_b, I_c, I_d be on this circumsphere are

$$-c^2S_aS_b - b^2S_aS_c - a'^2S_aS_d + a^2S_bS_c + b'^2S_bS_d + c'^2S_cS_d = 0 \tag{2}$$

$$-c^2S_aS_b + b^2S_aS_c + a'^2S_aS_d - a^2S_bS_c - b'^2S_bS_d + c'^2S_cS_d = 0 \tag{3}$$

$$c^2S_aS_b - b^2S_aS_c + a'^2S_aS_d - a^2S_bS_c + b'^2S_bS_d - c'^2S_cS_d = 0 \tag{4}$$

$$c^2S_aS_b + b^2S_aS_c - a'^2S_aS_d + a^2S_bS_c - b'^2S_bS_d - c'^2S_cS_d = 0. \tag{5}$$

Linear combinations of (2)–(5) give

$$\frac{S_a}{ab'c'} = \frac{S_b}{a'bc'} = \frac{S_c}{a'b'c} = \frac{S_d}{abc}. \tag{6}$$

* Translated from the French by W. E. Byrne, Virginia Military Institute.

From the relations $abc=4R_aS_a, \cdots$, and (6) it follows that

$$R_a = R_b = R_c = R_d, \tag{7}$$

and the circumcenter and incenter of T coincide. Hence T is isosceles [1].

Conversely, if T is isosceles, the points I_a, I_b, I_c, I_d are on the circumsphere of T [2]. Or (1) may be used with $a=a'$, $b=b'$, $c=c'$, *etc.*

Another proof. For an arbitrary tetrahedron T the incenter and the centers of the seven escribed spheres form with the given tetrahedron a desmic system [3]. According to a known theorem [4] if the points I_a, I_b, I_c, I_d are on the circumsphere of T, then the lines AI_a, BI_b, CI_c, DI_d meet at the incenter I, which coincides with the point L whose normal coordinates are proportional to R_a, R_b, R_c, R_d. Thus (7) holds. Consequently T is isosceles and conversely.

COROLLARY. *The tetrahedron $T'\equiv I_aI_bI_cI_d$ is isosceles if T is isosceles, and corresponding edges of these two tetrahedrons meet in pairs at their midpoints.*

Proof. T and T' are symmetric with respect to the center of the circumsphere of T. The circumcenter coincides with their common centroid [5].

COROLLARY. *The tetrahedrons I_aBCD, I_bCDA, I_cDAB, I_dABC are trirectangular at I_a, I_b, I_c, I_d and conversely.*

Proof. A given tetrahedron T may be bordered on the outside (inside) by trirectangular tetrahedrons if and only if all its faces are right triangles and this takes place when T is isosceles. The points I_a, I_b, I_c, I_d are diametrically opposite on the circumsphere of T to the vertices A, B, C, D. Let R be the radius of the circumsphere of T. Then

$$\begin{aligned}\overline{I_dB}^2 + \overline{I_dC}^2 &= \overline{I_dD}^2 - \overline{DB}^2 + \overline{I_dD}^2 - \overline{DC}^2 \\ &= 8R^2 - b'^2 - c'^2 = 8R^2 - b^2 - c^2 = a^2;\end{aligned}$$

and likewise,

$$\overline{I_dC}^2 + \overline{I_dA}^2 = b^2, \qquad \overline{I_dA}^2 + \overline{I_dB}^2 = c^2$$

since it is known [6] that $8R^2=a^2+b^2+c^2$.

Conversely, if I_dABC is trirectangular at I_d, its altitude drawn from I_d goes through the orthocenter H of the triangle ABC; H is the point of contact of the sphere (I_d) with the face ABC of T [7]. According to a known theorem [8] the inscribed sphere (I) touches the face ABC at the isogonal conjugate of H, *i.e.*, at the circumcenter of ABC. The same conclusions hold for the tetrahedrons I_aBCD, I_bCDA, I_cDAB, so the spheres $ABCD$, (I) are concentric and T is isosceles.

2. THEOREM. *In an arbitrary tetrahedron $T\equiv ABCD$ the orthogonal projections on the planes BCD, CDA, DAB, ABC of the isogonal conjugates A_2, B_2, C_2, D_2 of the vertices A_1, B_1, C_1, D_1 of the trirectangular tetrahedrons A_1BCD, B_1CDA, C_1DAB, D_1ABC which border T on the outside (inside) are the vertices of four tri-*

rectangular tetrahedrons. (Pedal tetrahedrons of A_2, B_2, C_2, D_2 with respect to T).

Proof. If we designate, as in plane geometry [9], by A_3, B_3, C_3, D_3 the orthogonal projections of D_2 on the planes BCD, CDA, DAB, ABC, the lines D_1A, D_1B, D_1C are perpendicular to the planes $B_3C_3D_3$, $C_3D_3A_3$, $D_3A_3B_3$ which are therefore perpendicular in pairs. Hence tetrahedron $D_3A_3B_3C_3$ is trirectangular at D_3. The same argument applies to the others.

COROLLARY. *The pedal tetrahedrons of the centers I_a, I_b, I_c, I_d with respect to the isosceles tetrahedron T are trirectangular at the orthocenters of the faces BCD, CDA, DAB, ABC and conversely.*

Proof. In this case I_a, I_b, I_c, I_d coincide with their isogonal conjugates with respect to T and the pedal tetrahedrons of these points are trirectangular at the orthocenters of the faces BCD, CDA, DAB, ABC. Conversely, if the pedal tetrahedron of I_d is trirectangular at H, then I_dABC is trirectangular at I_d and the conclusion follows from the corollary of paragraph 1.

References

1. N. A. Court, Modern Pure Solid Geometry, p. 97, art. 304.
2. *Ibid*, p. 100, art. 313.
3. *Ibid*, p. 235, art. 722 (definition of desmic system).
4. V. Thébault, Mathesis, t. 56, question 3370 and t. 61, p. 324
5. N. A. Court, *loc. cit.*, p. 95, art. 298; p. 100, art. 313.
6. *Ibid*, p. 102, ex. 18.
7. *Ibid*, p. 100, art. 315.
8. J. Neuberg, Mémoire sur le Tétraèdre, 1884, p. 23.
9. R. A. Johnson, Modern Geometry, p. 153, art. 213.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

L'HÔPITAL'S RULE AND EXPANSION OF FUNCTIONS IN POWER SERIES

M. R. SPIEGEL, Rensselaer Polytechnic Institute

When a student is first introduced to the concept of expansion of functions in power series, it is instructive to provide methods for evaluating the coefficients preliminary to using the conventional one of computing successive derivatives according to the rules of Maclaurin and Taylor. One such method, which will be described here, is useful when the student has had L'Hôpital's rule for the eval-

uation of indeterminate forms. For illustrative purposes the case of the expansion of $\sin x$ in powers of x will be discussed.

One begins with the customary assumption that

$$\sin x = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \tag{1}$$

is to be an identity in x. It is desired to determine what possible values of the a's are consistent with this assumption.

To evaluate a_0 let $x=0$. If (1) is to be an identity it follows that a_0 ought to be equal to zero. Hence

$$\sin x = a_1x + a_2x^2 + a_3x^3 + \cdots . \tag{2}$$

Dividing both sides of (2) by x yields

$$\frac{\sin x}{x} = a_1 + a_2x + a_3x^2 + \cdots . \tag{3}$$

Taking the limit of both sides of (3) as $x \to 0$ it follows that $a_1 = 1$. Hence

$$\frac{\sin x}{x} = 1 + a_2x + a_3x^2 + \cdots , \tag{4}$$

or

$$\frac{\sin x - x}{x} = a_2x + a_3x^2 + \cdots . \tag{5}$$

Dividing by x,

$$\frac{\sin x - x}{x^2} = a_2 + a_3x + \cdots . \tag{6}$$

Taking the limit of both sides of (6) as $x \to 0$, making use of L'Hôpital's rule in the evaluation of the indeterminate form on the left, we find $a_2 = 0$. Hence

$$\frac{\sin x - x}{x^2} = a_3x + a_4x^2 + \cdots . \tag{7}$$

Dividing by x and letting $x \to 0$ using L'Hôpital's rule again we have $a_3 = -\frac{1}{6}$ and so

$$\frac{\sin x - x}{x^3} = -\frac{1}{6} + a_4x + a_5x^2 + \cdots , \tag{8}$$

i.e.,

$$\frac{x^3 + 6 \sin x - 6x}{6x^3} = a_4x + a_5x^2 + \cdots \tag{9}$$

or

$$\frac{x^3 + 6 \sin x - 6x}{6x^4} = a_4 + a_5 x + \cdots, \tag{10}$$

from which the limit as $x \to 0$ gives $a_4 = 0$. A continuation of the procedure yields $a_5 = 1/120$. It is then found that

$$\frac{x^3 + 6 \sin x - 6x}{6x^5} = \frac{1}{120} + a_6 x + a_7 x^2 + \cdots. \tag{11}$$

In a similar way further coefficients may be found. By stopping at any desired place as, for example, at (11) and solving for $\sin x$, or by using the coefficients already found in the equation (1), the first few terms of the series are obtained. In this case the result is

$$\begin{aligned} \sin x &= x - \frac{x^3}{6} + \frac{x^5}{120} + a_6 x^6 + a_7 x^7 + \cdots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + a_6 x^6 + a_7 x^7 + \cdots \end{aligned} \tag{12}$$

from which the general pattern is indicated but, of course, not proved.

The author has found by approaching the problem in this way that students greatly welcome (and consequently remember better) the conventional method which is so much less tedious. The above procedure is useful also in that it provides a probably needed review of indeterminate forms and L'Hôpital's rule which by this time may have disappeared from the student's mathematical vocabulary. It also prepares him to see later a possible application of series in the evaluation of indeterminate forms, thus shortening the labor of applying L'Hôpital's rule too frequently in certain cases.

In addition this approach enables the student to realize that there may be many different ways of evaluating the coefficients in the power series expansion of a given function. In fact one possible method which could be suggested is the somewhat natural approach of letting x equal certain values on both sides of (1), such as for example $x = \pi/2^n$, $n = 1, 2, 3, \cdots$ for which $\sin x$ may always be found exactly. In this manner, the concept of an infinite number of equations involving an infinite number of unknowns and consequently determinants of infinite order could be mentioned as possibilities for further investigation.

At this stage it is even a good idea to inquire as to the possibility of expansion of functions into series other than power series, perhaps presenting just a brief mention of Fourier series.

It turns out that such remarks as given above often provide for some lively discussion in class and seem to add more to the meaning of mathematics and consequently to the better understanding of mathematics in general. It is certainly worth the little time spent.

THE CIRCUMSCRIBED CIRCLE

ARTHUR PORGES, Los Angeles City College

A common exercise in most analytic geometry textbooks is the determination of the equation of the circle circumscribing the triangle formed by three given straight lines. The method implied usually involves first finding the points of intersection of the lines, after which those points are used with some standard device like a determinant.

An approach suggested by the device often used for a conic through five given points, however, makes pairwise solution of the three linear equations unnecessary, and has some interest as a classroom variant.

Given the three straight lines

$$L_i \equiv a_iX + b_iY + c_i = 0, \qquad (i = 1, 2, 3), \tag{1}$$

we construct the function

$$U(X, Y) \equiv L_1L_2 + hL_1L_3 + kL_2L_3, \tag{2}$$

where h and k are parameters to be determined later.

Obviously, $U(X, Y)=0$ is satisfied by the points of intersection of the lines (1). Further, $U(X, Y)$ is a second degree function of X and Y of the form

$$\begin{aligned} U(X, Y) \equiv {} & A(h, k)X^2 + B(h, k)XY + C(h, k)Y^2 + D(h, k)X \\ & + E(h, k)Y + F(h, k). \end{aligned} \tag{3}$$

If we now impose the conditions

$$A(h, k) = C(h, k), \tag{4}$$

and

$$B(h, k) = 0, \tag{5}$$

it follows that the values of h and k derived from the simultaneous solution of the linear system (4) and (5) will reduce $U(X, Y)=0$ to the equation of the desired circle.

THE FUNDAMENTAL THEOREM OF THE DIFFERENTIAL CALCULUS

W. R. RANSOM, Tufts College

That every equation of a certain type has a root, and that the limit of a certain type of sum is an integral, are agreed to be *the* Fundamental Theorems of Algebra and of the Integral Calculus. What is there entitled to that designation in the Differential Calculus?

It should be something about differentials, since we are talking about the differential calculus. In spite of the diversity of opinion about these curious symbols or quantities (see recent controversies in this MONTHLY, vol. 58, p. 336 (1951), and vol. 59, pp. 392–406 (1952)), there seems to be a fact about them of

wide generality that is universally exhibited. This fact is usually presented in a paragraph with the phrase "function of a function" in its title: it is embodied in this equation,

$$\operatorname{Limit}_{\Delta z \to 0} \frac{\Delta y}{\Delta z} = \frac{dy}{dz}.$$

The qualifications usually specified for this require that y shall be a function of x, that x shall be a function of z, and that certain derivatives exist and are not zero. Instead of the dy/dz, we find sometimes the product $(dy/dx)(dx/dz)$ or the quotient $(dy/dx)/(dz/dx)$.

If we wish to make a very useful choice of what to take as the fundamental theorem of the differential calculus, we do well to reconsider what the differential calculus is about for the stage at which it is first taught.

The student comes to the differential calculus from a study of analytical geometry and physics. In these subjects he is familiar with quantities whose measurements yield varying values, but for which *natura non agit per saltum.*

With average rates he has long been familiar, and the first task of the calculus is to familiarize him with the new concept of an "instantaneous rate." The average rate is a convenience in everyday life: it represents no reality in nature, but is merely an expedient in simplifying calculations. The limiting values, however, have important physical meanings: the overturning of a car on a curve does not depend upon its average speed, $\Delta s/\Delta t$, but upon a critical value, $ds/dt = \sqrt{F \cdot R}$, (appropriate units being assumed), where F is the force required for overturning and R the radius of the curve.

The first chapter in many textbooks develops Cauchy's idea of "function," and introduces freakish possibilities, which will be quite exceptional and usually ignored. We have been told that it is common practice to omit this chapter and proceed to the next. This omission is good practice. It is well to build up the idea of differentiation not so much on Cauchy's abstraction as upon the ideas which the student already understands, and concepts which will be readily applicable to the problems for which he is to use the differential calculus.

In his problems, he will be concerned with groups of quantities, some constant, others variable. These quantities are connected by equations with which he is familiar, $y=x^2$, $A=Bh$, $pv=c$, *etc.* Among these variables there is no "independent variable": independence is an attribute imposed *temporarily* by the mathematician—there is nothing in nature or geometry that dictates its choice. The idea of independence should be relegated to the background as soon as possible. Instead of a group of n functions of an independent variable, we have $n+1$ variables, connected by M equations: from these equations M differential equations can be obtained, and from these, $M(n+1)$ equations can be obtained (by division) that connect the $n(n+1)$ possible rates with the variables.

In such a group of variables we may consider two states, and by subtractions determine relations involving the increments (positive, negative, or zero) by

which the variables are affected. Among these variables we may arbitrarily select one variable, x, and compute the derivatives of the other variables with respect to x. Selecting at pleasure a number to be denoted by "dx" (in no way dependent upon the size of Δx), called "differential of x," we may express the derivatives of all the other variables as fractions with dx as denominator, and denote their numerators by da, db, dc, *etc.* This gives us as definitions of the other differentials

$$da/dx = \operatorname*{Limit}_{\Delta x \to 0} (\Delta a/\Delta x), \qquad db/dx = \operatorname*{Limit}_{\Delta x \to 0} (\Delta b/\Delta x), \quad \textit{etc.}$$

The differentials having been defined in this way, by means of an arbitrarily chosen variable, we are now able to abandon the distinction between the variables, and make an important and useful statement about any one of the instantaneous rates. This statement deserves to be called *The Fundamental Theorem of the Differential Calculus.*

With the implied reservation that there can be no division by zero, the fundamental theorem is that whatever variables, y and z, are selected from the list, a, b, c, *etc.*, we shall have the relation that

$$\operatorname*{Limit}_{\Delta z \to 0} \frac{\Delta y}{\Delta z} = \frac{dy}{dz}.$$

The proof is very simple, requiring only an understanding about the limit of a product, and is so familiar as given in paragraphs about the function of a function, that we need not give it here.

This theorem is of such a fundamental nature, and of such frequent and useful applicability, that it should be brought to the very beginning of the study of the differential calculus, and recognized as of importance equal to that of the fundamental theorem of the integral calculus.

TRANSFORMATION OF STANDARD INTEGRALS

V. PUNGA, Rensselaer Polytechnic Institute

In every textbook of integral calculus we can find the tables of standard integrals. The problem of integration consists in reducing the given integral to one of the standard type.

Probably not everybody is aware of the fact that there exist transformations transforming any one of the standard integrals into any other one. The transformation formulas can be found in the following way:

If $\int f_1(v)dv = \phi_1(v)$ and $\int f_2(u)du = \phi_2(u)$, then, in order to transform $\int f_1(v)dv$ into $\int f_2(u)du$, we have to solve $\phi_1(v) = \phi_2(u)$ for v in terms of u, say $v = \psi(u)$, which is the required transformation. The transformation is impossible if we can not solve $\phi_1(v) = \phi_2(u)$ for v in terms of u (in elementary functions).

For example, in order to transform $\int \sec v \, dv (= \ln (\sec v + \tan v))$ into $\int e^u du (= e^u)$, setting $C = 0$, we set up: $\ln (\sec v + \tan v) = e^u$ or $\sqrt{1+\tan^2 v} + \tan v$

$=e^{(e^u)}$. Solving this equation for v, we obtain

$$v = \arctan \tfrac{1}{2}(e^{e^u} - e^{-e^u}).$$

Hence,

$$dv = \frac{2e^u du}{e^{e^u} + e^{-e^u}}, \qquad \sec v = \frac{e^{e^u} + e^{-e^u}}{2}$$

and therefore:

$$\int \sec v dv = \int \frac{e^{e^u} + e^{-e^u}}{2} \cdot \frac{2e^u du}{e^{e^u} + e^{-e^u}} = \int e^u du.$$

As a curiosity I suggest the following method of integration (without taking it too seriously). Let the students memorize only one standard integral, say $\int e^u du = e^u + C$, and teach them how to reduce any other integral to this one. In this way we can reduce the table of standard integrals to only one formula.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1166. *Proposed by Leon Bankoff, Los Angeles, Calif.*

Let DE be a variable chord perpendicular to diameter AB of a given circle (O). The maximum circle (ω_0) inscribed in the smaller segment, DEB, touches chord DE in C. The circle (ω_1) is tangent to (ω_0), (O), and DC, and another circle (ω_2) is tangent to (ω_1), (O), and DC. Find the ratio BC/CA for which the radius of circle (ω_2) is a maximum.

E 1167. *Proposed by C. D. Olds, San Jose State College*

Prove by mathematical induction that if a is a real number $\neq 1$, n a positive integer, then

$$\alpha_n = \frac{a^{2n+2} - 1}{a(a^{2n} - 1)} > \frac{n+1}{n}.$$

E 1168. *Proposed by R. R. Phelps, U.C.L.A.*

In an analogy with perfect numbers, let us define a *perfect triangle* as one whose integer valued sides add up to twice its area. An example is the (3, 4, 5) triangle. Find all perfect triangles.

E 1169. *Proposed by Joseph Lehner, Los Alamos Laboratory*

Let $0<x_1<x_2<\cdots$ and let $\sum_{n=1}^{\infty} x_n^{-h}$ converge. Then, if $0<h\leqq 1$ and $\alpha>0$, we have $x_{n+1}-x_n>\alpha$ for infinitely many indices n. If $h>1$, there are sequences $\{x_n\}$ with $x_{n+1}-x_n\to 0$.

E 1170. *Proposed by Viktors Linis, University of Ottawa*

Show that there exists a centrally-symmetric hexagon inscribed in any closed convex curve such that the ratio of the respective areas is at least 2/3.

SOLUTIONS

Calculating an Average Value

E 1136 [1954, 642]. *Proposed by W. A. Bowers, University of North Carolina*

An elementary physics text, wishing to avoid using calculus, and also hoping (unsuccessfully!) to avoid the appearance of pulling the answer out of a hat, gives the following recipe for calculating the work done in moving a positive unit charge from r_1 to r_2 against the attraction of a negative unit charge at the origin. "Since the force varies with distance, we cannot simply multiply 'the force' by the distance; instead we must take the geometric mean of the initial and final values of the force, and multiply it by the distance." Noting with amazement that this does indeed work, we ask: what is the most general function for which this is true, that is, for which the average value over an arbitrary interval equals the geometric mean of the values at the end points?

Solution by L. A. Ringenberg, Eastern Illinois State College. Let $F(x)$ have a differentiable positive square root $f(x)$ and satisfy the condition of the problem. Then

$$\int_x^y [f(t)]^2 dt = f(x)f(y)(y-x).$$

Taking partial derivatives we get

$$[f(y)]^2 = f(x)f'(y)(y-x) + f(x)f(y)$$

and

$$-[f(x)]^2 = f'(x)f(y)(y-x) - f(x)f(y).$$

From these relations we obtain

$$[f(y)-f(x)]/(y-x) = f(x)f'(y)/f(y) = f(y)f'(x)/f(x),$$

whence

$$f'(x)/[f(x)]^2 = f'(y)/[f(y)]^2,$$

and, since x and y are arbitrary,

$$f'(x)/[f(x)]^2 = c, \qquad \text{a constant.}$$

Therefore

$$F(x) = [f(x)]^2 = (hx + k)^{-2}, \qquad h \text{ and } k \text{ arbitrary.}$$

Conversely, we may easily verify by substitution that any function of the form $F(x) = (hx+k)^{-2}$ satisfies the desired relation.

Also solved by P. M. Anselone, H. E. Bray, G. B. Charlesworth, I. A. Dodes, Virginia Hanly, W. W. Hooker, E. S. Keeping, P. G. Kirmser, C. S. Ogilvy, Azriel Rosenfeld, J. P. Scholz, R. P. Tapscott, Chih-yi Wang, David Zeitlin, and the proposer. Late solutions by M. S. Klamkin, C. F. Pinzka, and D. C. Russell.

Editorial Note. It is to be observed that the attraction function of the proposal is essentially already the most general one.

An Easy Inequality

E 1137 [1954, 642]. *Proposed by Simon Green, Philander Smith College, Arkansas*

Let $N = 1+1/2+1/3+ \cdots +1/n$. Prove that $e^N > n+1$.

I. *Solution by N. J. Fine, University of Pennsylvania.*

$$\begin{aligned} e^N &= e^1 e^{1/2} e^{1/3} \cdots e^{1/n} \\ &> (1 + 1)(1 + 1/2)(1 + 1/3) \cdots (1 + 1/n) \\ &= (2)(3/2)(4/3) \cdots (n + 1)/n = n + 1. \end{aligned}$$

II. *Solution by W. E. Briggs, University of Colorado.* The inequality holds for $n = 1$. Assume it also holds for $n = 2, 3, \cdots, k$. Then

$$e^{1+1/2+\cdots+1/k} > k + 1$$

and

$$\begin{aligned} e^{1+1/2+\cdots+1/(k+1)} &> e^{1/(k+1)}(k + 1) \\ &> [1 + 1/(k + 1)](k + 1) \\ &= k + 2, \end{aligned}$$

and the inequality holds for all n by induction.

III. *Solution by Bernard Greenspan, Drew University.* Since

$$N = 1 + 1/2 + \cdots + 1/n > \int_1^{n+1} dx/x = \ln (n + 1),$$

we have $e^N > n+1$.

Also solved by A. N. Aheart, Arlo Anderson, P. M. Anselone, J. W. Baldwin, Leon Bankoff, P. T. Bateman, H. W. Becker, Julian Braun, K. A. Bush, C. N. Campopiano, W. B. Carver, N. A. Childress, R. J. Cormier, Hüseyin Demir, I. A. Dodes, Edgar Dougherty, F. J. Duarte, H. M. Feldman, Edward Fleisher, Calvin Foreman, J. E. Freund, Joyce Friedman, Ruth Frisch, Zoffmann Fudita, Laurence Glasser, A. M. Glicksman, Sidney Glusman, E. S. Grable and J. W. Sawyer (jointly), Nathaniel Grossman, D. S. Greenstein, Vern Hoggatt, W. W. Hooker, P. F. Hultquist, A. R. Hyde, P. G. Kirmser, Sidney Kravitz, M. J. Mansfield, D. C. B. Marsh, W. E. Mientka, J. D. Miller, Leo Moser, T. F. Mulcrone, J. B. Muskat, C. S. Ogilvy, C. D. Olds, S. Parameswaran, F. D. Parker, L. L. Pennisi, W. J. Pervin, R. B. Plymale, O. J. Ramler, B. E. Rhoades, L. A. Ringenberg, Azriel Rosenfeld, David Sachs, C. M. Sandwick, Sr., Frank Saunders, M. Schwartz, Berthold Schweizer, F. A. Sherk, Nathan Shklov, Augustus Sisk, O. E. Stanaitis, D. D. Strebe, D. R. Sudborough, A. V. Sylwester, R. P. Tapscott, J. A. Tierney, Alan Wayne, Roy Westwick, Howard Wicke, K. B. Williams, David Zeitlin, and the proposer. Late solutions by A. E. Anderson, M. S. Klamkin, C. F. Pinzka, and D. C. Russell.

Sharper bounds for e^N were given by Bush and Rhoades. Thus Bush showed that $e^N \geqq e(n+1)/2$, equality holding only if $n=1$, and Rhoades showed that $e^N > n+17/10$.

Bisosceles and Trisosceles Triangles

E 1138 [1954, 642]. *Proposed by J. P. Ballantine, University of Washington*

For any triangle prove that: (1) if $B=2A$, then $b^2=a^2+ac$, (2) if $B=3A$, then $b^3-ab^2-a^2b-ac^2+a^3=0$.

I. *Solution by D. C. B. Marsh, Texas Technological College.* (1) Since $B=2A$, $C=\pi-3A$ and (by a Mollweide equation)

$$(a+c)/b = \cos(2A-\pi/2)/\sin A = \sin 2A/\sin A = b/a.$$

The result follows at once.

(2) Here $C=\pi-4A$ and (again by a Mollweide equation)

$$c/(b-a) = \cos(\pi/2-2A)/\sin A = \sin 2A/\sin A$$
$$= 2\cos A = (b^2+c^2-a^2)/bc.$$

The result follows at once.

II. *Solution by Joyce Friedman, ACF Electronics, Alexandria, Va.* (1) Draw BD, the bisector of B. Then triangles BDC and ABC are similar, and

$$a/b = BD/c = (b-AD)/a.$$

Since triangle ABD is isosceles, $AD=BD$, whence

$$AD = ac/b \quad \text{and} \quad a/b = (b-ac/b)/a.$$

That is, $b^2=a^2+ac$.

(2) Draw BD such that angle $ABD=2A$, angle $DBC=A$. Then triangles BDC and ABC are similar, and

$$BD/c = a/b = (b - AD)/a.$$

By part (1), $(AD)^2=(BD)^2+(BD)c$. But $BD=ac/b$, and therefore

$$a/b = (b - \sqrt{(ac/b)(ac/b + c)})/a.$$

Simplifying this gives the desired result, $b^3-ab^2-a^2b-ac^2+a^3=0$.

Also solved by A. N. Aheart, R. V. Andree's freshman mathematics class, Norman Anning, Leon Bankoff, H. H. Berry, Margaret Blue, W. B. Carver, R. L. Caskey, G. B. Charlesworth, P. L. Chessin, N. A. Childress, Shannon Clark, K. W. Crain, J. E. D'Atri, Hüseyin Demir, I. A. Dodes, F. J. Duarte, A. L. Epstein, H. M. Feldman, Russell Godard, Bernard Greenspan, Vern Hoggatt, Douglas Holdridge, W. W. Hooker, Raymond Huck, P. F. Hultquist, A. R. Hyde, John Jones, Jr., Sam Kravitz, Sidney Kravitz, R. A. Miller, George Millman, Lawrence Miner, J. B. Muskat, C. S. Ogilvy, S. Parameswaran, M. J. Pascual, W. O. Pennell, P. W. A. Raine, B. E. Rhoades, L. A. Ringenberg, Azriel Rosenfeld, F. W. Saunders, J. W. Sawyer, F. A. Sherk, Augustus Sisk, O. E. Stanaitis, E. P. Starke, J. A. Tierney, Alan Wayne, Roy Westwick, Dale Woods, Roscoe Woods, Hazel S. Wilson, David Zeitlin, and the proposer. Late solutions by M. S. Klamkin, Josef Langr, C. F. Pinzka, and D. C. Russell.

Editorial Note. Miller pointed out that this problem is the converse of his Problem 123, *National Mathematics Magazine*, vol. 2 (1936), p. 58.

Miss Friedman's method (solution II above) may be successively used to obtain relations for triangles having $B=4A$, $5A$, $\cdots$. This raises the problem of finding a general formula, involving the sides of the triangle, equivalent to $B=nA$, where n is a positive integer. In problem E 620 [1945, 46], Wayne pointed out that the identity

$$\sin (n + 1)A = r \sin nA - \sin (n - 1)A,$$

where $r=2 \cos A$, together with the law of sines, yields the equality

$$\frac{b}{c} = \frac{1}{r - \dfrac{1}{r - \dfrac{1}{r - \cdots \dfrac{1}{r}}}}$$

(to n components), where $B=nA$ and $r=(b^2+c^2-a^2)/bc$.

The relation of part (1) is interestingly applicable to the three familiar triangles with angles $(A, A, 2A)$, $(A, 2A, 2A)$, $(A, 2A, 4A)$.

An Identity Projectivity

E 1139 [1954, 642]. *Proposed by N. A. Court, University of Oklahoma*

Three collinear points P, Q, R are marked on the sides BC, CA, AB of a triangle ABC. Starting with an arbitrary point X of the line BC, the following

points are constructed successively:

$$Y = (XR, CA), \quad Z = (YP, AB), \quad X' = (ZQ, BC);$$
$$Y' = (X'R, CA), \quad Z' = (Y'P, AB), \quad X'' = (Z'Q, BC).$$

Show that points X and X'' coincide.

I. *Solution by Chih-yi Wang, University of Minnesota.* It is not necessary to assume that P, Q, R are collinear. By Menelaus' theorem we have the following six relations:

$$(1) \quad (AR/RB)(BX/XC)(CY/YA) = -1,$$
$$(2) \quad (AZ/ZB)(BP/PC)(CY/YA) = -1,$$
$$(3) \quad (AZ/ZB)(BX'/X'C)(CQ/QA) = -1,$$
$$(4) \quad (AR/RB)(BX'/X'C)(CY'/Y'A) = -1,$$
$$(5) \quad (AZ'/Z'B)(BP/PC)(CY'/Y'A) = -1,$$
$$(6) \quad (AZ'/Z'B)(BX''/X''C)(CQ/QA) = -1,$$

Forming the product of the three ratios (1) to (2), (3) to (4), (5) to (6), we obtain

$$BX/XC = BX''/X''C,$$

whence X and X'' coincide.

II. *Solution by Hüseyin Demir, Zonguldak, Turkey.* From

$$X \overset{R}{\underset{\wedge}{=}} Y \overset{P}{\underset{\wedge}{=}} Z \overset{Q}{\underset{\wedge}{=}} X' \overset{R}{\underset{\wedge}{=}} Y' \overset{P}{\underset{\wedge}{=}} Z' \overset{Q}{\underset{\wedge}{=}} X''$$

we get the projectivity $X \overset{-}{\wedge} X''$, in which B, C, P are self-corresponding elements. Hence all the points of BC are self-corresponding elements. It is to be noted that the property holds even if the points P, Q, R are not collinear.

III. *Solution by W. B. Carver, Cornell University.* Using a homogeneous coordinate system, let the vertices A, B, C be the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and let the points P, Q, R be, respectively, $(0, a, 1)$, $(1, 0, b)$, $(c, 1, 0)$, and let X be $(0, x, 1)$. We then find successively $Y(-cx, 0, 1)$, $Z(cx, a, 0)$, $X'(0, a, -bcx)$, $Y'(a, 0, bx)$, $Z'(1, -bx, 0)$, $X''(0, x, 1)$. Thus X'' coincides with X, whether the points P, Q, R are collinear or not.

IV. *Solution by O. J. Ramler, The Catholic University of America.* A proof can be given based on Sylvester's residuation theorem, "All cubics through eight points pass through a unique ninth point."

Let $p=0$, $q=0$, $r=0$, $p'=0$, $q'=0$, $r'=0$ be the equations of the lines PYZ, $QX''Z'$, $RX'Y'$, RXY, $PY'Z'$, $QX'Z$, and let $a=0$, $b=0$, $c=0$ be the equations of the lines BC, CA, AB. Then $pqr=0$, $p'q'r'=0$, $abc=0$ are three composite cubics each passing through the eight points P, Q, R, X', Y', Z', Y, Z. The first

two cubics also pass through the ninth point W common to $QX''Z'$ and RXY, whence the third cubic also must pass through this ninth point. But W cannot lie on $b=0$ or on $c=0$, since these lines contain QY and $Z'R$ respectively. Hence W lies on $a=0$, and X, X'', W all coincide.

Note that we have not assumed that P, Q, R are collinear.

Also solved by R. L. Caskey, G. B. Charlesworth, K. W. Crain, Carolyn A. Denslow and Claire A. Pinney (jointly), N. J. Fine, D. C. B. Marsh, Geoffrey Mathews, T. F. Mulcrone, M. W. Oliphant, S. Parameswaran, David Sacks, F. A. Sherk, Roy Westwick, Roscoe Woods, and the proposer. Late solutions by Josef Langr and C. F. Pinzka.

Related Sequences

E 1140 [1954, 642]. *Proposed by Albert Wilansky, Lehigh University*

Let sequences $\{x_n\}$, $\{y_n\}$ be called related if $\sum_{k=1}^{n}(x_k-y_k)$ is a bounded function of n. For example, $\{(-1)^n\}$ and $\{0\}$ are related. Is there a bounded sequence not related to any convergent sequence?

Solution by A. Cesàro, University of Wisconsin. If $\{x_n\}$ and $\{y_n\}$ are "related," then $(1/n)\left|\sum_1^n (x_k-y_k)\right| \to 0$. That is, $C_1\text{-lim}\,(x_n-y_n)=0$. If $\{y_n\}$ is convergent, it is C_1 summable, and it is therefore necessary that $\{x_n\}$ be C_1 summable. Thus no bounded sequence $\{x_n\}$ which is non-C_1 summable can be "related" to any convergent sequence $\{y_n\}$. Such will be the case, for example, for the sequence

$$1, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, \cdots.$$

Also solved by K. A. Bush, N. J. Fine, M. S. Klamkin, C. F. Pinzka, George Piranian, and H. H. Wicke.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4638. *Proposed by Paul Erdös, University of Notre Dame*

Let $k>2$. Does the following equation have any solutions in integers:

$n(n+1)\cdots(n+k-1) = 2m(m+1)\cdots(m+k-1), \quad m+k-1 < n?$

For $k=2$ there are infinitely many solutions, easily determined.

4639. *Proposed by L. B. Rall, Oregon State College*

Given a set of distinct numbers $a_0, a_1, \cdots$, such that $a_j \neq a_k$ if $j \neq k$, prove that, for all positive integers n,

$$\sum_{j=0}^{n} \prod_{k=0, k\geq j}^{n} \frac{(a_k - a_{n+1})}{(a_k - a_j)} = 1.$$

4640. *Proposed by M. L. Keedy, University of Nebraska*

Let H be a proper subgroup of a group G and $\overline{H}$ its complement in G. Show that $H' = \overline{H}\,\overline{H}$ is a normal subgroup of G, and that it equals either G or H. Characterize those subgroups for which $H' = H$.

4641. *Proposed by E. P. Starke, Rutgers University*

Given N, however large, prove there exist primes P such that each of the numbers 1, 2, 3, $\cdots$, N is a quadratic residue (mod P).

4642. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, New York*

Let $a_2, a_3, \cdots, a_n$ be arbitrary. Prove there must be an x in (0, 1) such that

$$\left| x + a_2x^2 + \cdots + a_nx^n \right| \geq \frac{1}{n} \tan \frac{\pi}{4n}.$$

SOLUTIONS

Powers of Infinite Matrices

4576 [1954, 126]. *Proposed by G. A. Garreau, Northampton Polytechnic College, London, England*

Construct an infinite matrix A such that A^{m-1} exists, whereas A^m does not.

Solution by the Proposer. Let m be positive, and $p<m$. Let $c>1$. Define the infinite matrix $A^{(p)}$ as follows: all its elements are zero except those lying in the first row, the second column, or the principal diagonal. In addition $a_{1,1}^{(p)} = a_{2,2}^{(p)} = 0$, so that the first column and the second row consist entirely of zeros. The nonzero elements are

$$a_{1,2}^{(p)} = \sum_{i=3}^{\infty} c^{-i(m-p)}; \qquad a_{1,k}^{(p)} = c^{k(p-1)}, \qquad k = 3, 4, \cdots;$$

$$a_{n,2}^{(p)} = c^{-n(m-1-p)}; \qquad a_{n,n}^{(p)} = c^{np}, \qquad n = 3, 4, \cdots.$$

We now show that, if p, q are two numbers such that $p+q<m$, then $A^{(p)}\times A^{(q)}=A^{(p+q)}$. Writing $A^{(p)}\times A^{(q)}=B=(b_{n,k})$ we have

$$b_{1,2}=\sum_{k=1}^{\infty}a_{1,k}^{(p)}a_{k,2}^{(q)}=\sum_{k=2}^{\infty}c^{k(p-1)}c^{-k(m-1-q)}=\sum_{k=2}^{\infty}c^{-k(m-p-q)}=a_{1,2}^{(p+q)}.$$

If $k>2$,

$$b_{1,k}=\sum_{i=1}^{\infty}a_{1,i}^{(p)}a_{i,k}^{(q)}=a_{1,k}^{(p)}a_{k,k}^{(q)}=c^{k(p-1)}c^{kq}=a_{1,k}^{(p+q)}.$$

If $n>2$,

$$b_{n,2}=\sum_{i=1}^{\infty}a_{n,i}^{(p)}a_{i,2}^{(q)}=a_{n,n}^{(p)}a_{n,2}^{(q)}=c^{np}c^{-n(m-1-q)}=a_{n,2}^{(p+q)},$$

$$b_{n,n}=\sum_{i=1}^{\infty}a_{n,i}^{(p)}a_{i,n}^{(q)}=a_{n,n}^{(p)}a_{n,n}^{(q)}=c^{np}c^{nq}=a_{n,n}^{(p+q)}.$$

All further elements are zero. Hence $A^{(p)}\times A^{(q)}=A^{(p+q)}$, as required.

We now write A for $A^{(1)}$, and obtain by induction $A^{(p)}=A^{p}$, $p<m$, for p a positive integer, the product on the right being associative whenever it exists. We can define A^{p} as $A^{(p)}$ even if p is not an integer, and the index laws are obeyed if $p<m$.

However, $A^{(m)}$ does not exist, since $a_{1,2}^{(m)}$ is defined as the sum of an infinite series of 1's, and, *a fortiori*, $A^{(p)}$ does not exist if $p>m$.

Nested Neighborhoods in a Hausdorff Space

4577 [1954, 126]. *Proposed by Albert Wilansky, Lehigh University*

Given a point x in a Hausdorff space, does there exist a set of neighborhoods of x whose intersection contains only x, and such that of any two neighborhoods in the set one includes the other (*i.e.*, the neighborhoods are "nested")?

Solution by M. K. Fort, Jr., University of Georgia. It is possible that no such set of neighborhoods exists. Let A be an uncountable set, and let x be a point of A. We define a subset G of A to be open if and only if either G does not contain the point x or $A-G$ is a finite set. It is easy to verify that the resulting topology is Hausdorff.

Now suppose that there exists a "nested" set N of neighborhoods of x whose intersection contains only x. We let C be a countably infinite subset of $A-\{x\}$. For each point $c\in C$ there exists $G_c\in N$ such that $c\notin G_c$.

If $G\in N$, then there exists $c\in C\cap G$. It follows that $G\not\subset G_c$ and hence $G_c\subset G$. Thus the intersection of all of the neighborhoods in N is the same as the intersection of the sets G_c, $c\in C$. However, the intersection of the sets G_c, $c\in C$, contains all but a countable number of points of A and hence contains points other than x. This is a contradiction.

Also solved by H. D. Block, L. C. Dean, Jr., David Ellis, Leonard Gillman,

P. R. Halmos, J. R. Schoenfield and L. R. Ford, Jr., H. E. Vaughan, and L. E. Ward, Jr.

Canasta Hands

4578 [1954, 198]. *Proposed by N. S. Mendelsohn, University of Manitoba*

Find the number of essentially different eleven card Canasta hands which can be dealt from a 104 card pack. (We ignore red threes since they are always replaced in a hand. The pack consists of 8 aces, 8 twos, 4 threes, 8 fours, 8 fives, $\cdots$, 8 kings and 4 jokers. All cards of a given denomination are considered identical.)

I. *Solution by C. F. Pinzka, Educational Testing Service, Princeton, N. J.* The number of hands is the coefficient of x^{11} in

$$(1 + x + \cdots + x^4)^2(1 + x + \cdots + x^8)^{12} = (1 - x^5)^2(1 - x^9)^{12}(1 - x)^{-14},$$

where terms of the form $x^a x^b \cdots x^n$ are seen to correspond to hands containing $a, b, \cdots, n$ cards of the respective denominations.

Neglecting powers of x not involved, the desired result is the coefficient of x^{11} in $(1-2x^5-12x^9+x^{10})\sum\binom{k+13}{k}x^k$, which is

$$\binom{24}{11} - 2\binom{19}{6} - 12\binom{15}{2} + \binom{14}{1} = 2{,}440{,}634.$$

II. *Solution by Leo Moser, University of Alberta.* Let $C_r[a^i, b^j]$ denote the coefficient of x^r in

$$f(x) = (1 + x + x^2 + \cdots + x^a)^i(1 + x + x^2 + \cdots + x^b)^j.$$

The solution of the given problem is then

$$C_{11}[8^{12}, 4^2] = 2{,}440{,}634.$$

In "A problem in combinatory analysis," (*Trans. Roy. Sci. Can.* 48 (1953), pp. 21–26), the proposer gives the relatively efficient formula

$$C_r[a^i, b^j] = \sum_k \sum_m (-1)^{k+m}\binom{i}{k}\binom{j}{m}\binom{i+j+r-1-(a+1)k-(b+1)m}{r-(a+1)k-(b+1)m}.$$

A simple proof of this formula is given by the author, "On a combinatorial formula of Mendelssohn," (*Trans. Roy. Sci. Can.* 48(1953), p. 27).

Also solved by T. N. E. Greville, Bart Park, R. E. Shafer, and the Proposer.

Factors of e^z

4579 [1954, 199]. *Proposed by I. J. Schoenberg, University of Pennsylvania*

Let the relation

$$e^z = \left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right)$$

hold for $|z| < r$, where $a_n \geqq 0$, $b_n \geqq 0$ $(n = 0, 1, \cdots)$. Show that the two factors on the right side must be entire functions of the form

$$e^{az+c}, \qquad e^{bz-c}, \qquad (a \geqq 0,\ b \geqq 0,\ a + b = 1).$$

Solution by Leonard Carlitz, Duke University. We shall prove the following stronger result. Assume $\alpha_0 = \beta_0 = 1$,

$$\sum_{r=0}^{n} \binom{n}{r} \alpha_r \beta_{n-r} = 1, \qquad \alpha_n,\ \beta_n \geqq 0 \qquad (n = 1, 2, \cdots).$$

Then $\alpha_n = a^n$, $\beta_n = b^n$, $a \geqq 0$, $b \geqq 0$, $a + b = 1$. (The assumption $\alpha_0 = \beta_0 = 1$ involves no loss in generality.)

Proof. It follows immediately from the hypothesis that $0 \leqq \alpha_n \leqq 1$, $0 \leqq \beta_n \leqq 1$ for $n = 0, 1, 2, \cdots$. Therefore

$$f(z) = \sum_{n=0}^{\infty} \frac{\alpha_n z^n}{n!}, \qquad g(z) = \sum_{n=0}^{\infty} \frac{\beta_n z^n}{n!}$$

are entire functions of z such that

$$e^z = f(z)g(z).$$

Moreover

$$|f(z)| \leqq e^{|z|}, \qquad |g(z)| \leqq e^{|z|}.$$

Thus the functions $f(z)$, $g(z)$ are of order 1 at most and have no zeros; consequently by Hadamard's theorem (Titchmarsh, *Theory of Functions*, p. 250)

$$f(z) = e^{az}, \qquad g(z) = e^{bz},$$

where $a + b = 1$, $a \geqq 0$, $b \geqq 0$. This proves our theorem.

Also solved by M. Aissen, I. N. Baker, George Brauer, Paul Erdös and Michael Golomb, Viktors Linis, Edgar Reich, H. S. Shapiro, O. E. Stanaitis, and the Proposer.

Editorial Note. Erdös and Golomb show that the proposed conclusion follows from the weaker hypothesis

$$a \leqq \arg a_n \leqq A, \qquad b \leqq \arg b_n \leqq B, \qquad A + B - (a + b) < \pi.$$

Several contributors noted the probabilistic interpretation: If the convolution of two discrete probability distributions, $\sum_0^\infty a_n = 1$, $\sum_0^\infty b_n = 1$, is a Poisson distribution, then also the factor distributions are Poisson distributions. For the analogous result concerning the normal distribution see H. Cramer, Über eine Eigenschaft der normalen Verteilungsfunktion, *Math. Zeitschrift*, 41 (1936), pp. 405–414. See also H. Cramer, *Random Variables and Probability Distributions*, p. 52.

Series with Non-negative Coefficients

4580 [1954, 199]. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

For what real values of a does the power series expansion of the function $(1-ax+ax^2-x^3)^{-1}$ have all its coefficients non-negative?

Solution by L. Carlitz, Duke University. Put

$$(1 - ax + ax^2 - x^3)^{-1} = (f(x))^{-1} = \sum_{n=0}^{\infty} c_n x^n.$$

We shall prove the following

THEOREM. *$c_n \geqq 0$ for all $n \geqq 0$ if and only if $a = 0$ or $a \geqq 3$ or $a = 1+2\cos 2\pi/k$, where k is an integer > 2.*

Since $c_1 = a$, $c_2 = a^2 - a$, it is clear that $a \geqq 0$ is a necessary condition; indeed, if $a \neq 0$, it is necessary that $a \geqq 1$. Write

$$f(x) = (1 - x)(1 + (1 - a)x + x^2) = (1 - x)(1 - \beta x)(1 - \gamma x);$$

the factors are distinct unless $a = -1$ (which may be ignored) or $a = 3$. For $a = 3$ we get

$$(1 - x)^{-3} = \sum_{n=0}^{\infty} \frac{1}{2}(n + 1)(n + 2)x^n,$$

so that $c_n > 0$ for all $n \geqq 0$. Excluding this case we put

$$\frac{1}{f(x)} = \frac{A}{1 - x} + \frac{B}{1 - \beta x} + \frac{C}{1 - \gamma x}$$

and find (using $\beta + \gamma = a - 1$, $\beta\gamma = 1$) that

$$A = \frac{1}{(\beta - 1)(\gamma - 1)}, \qquad B = \frac{\beta^2}{(\beta - 1)(\beta - \gamma)}, \qquad C = \frac{\gamma^2}{(\gamma - 1)(\gamma - \beta)}.$$

It follows that

$$(1) \qquad c_n = A + B\beta^n + C\gamma^n = \frac{1}{(\beta - 1)(\gamma - 1)} + \frac{1}{\beta - \gamma}\left(\frac{\beta^{n+2}}{\beta - 1} - \frac{\gamma^{n+2}}{\gamma - 1}\right)$$

$$= \frac{1}{\beta - \gamma}\left(\frac{\beta^{n+2} - \beta}{\beta - 1} - \frac{\gamma^{n+2} - \gamma}{\gamma - 1}\right)$$

$$= \sum_{r=1}^{n+1} \frac{\beta^r - \gamma^r}{\beta - \gamma} = \sum_{r=0}^{n} (\beta^r + \beta^{r-1}\gamma + \cdots + \gamma^r).$$

The discriminant of the quadratic $1+(1-a)x+x^2$ is $(a-3)(a+1)$. We accordingly have two cases to consider: (i) $a>3$, (ii) $1<a<3$. In case (i) it is easily verified that β and γ are real and positive and therefore $c_n>0$ for all $n\geqq 0$.

In case (ii) we find that β and γ are complex, $|\beta| = |\gamma| = 1$. We put

$$u = e^{i\theta}, \qquad \beta = u^2, \qquad \gamma = u^{-2}, \qquad \beta + \gamma = 2\cos 2\theta = a - 1,$$

so that $0<\theta<\pi/4$. Then (1) reduces to

$$c_n = \frac{\sin (n+1)\theta \sin (n+2)\theta}{\sin\theta \sin 2\theta}.$$

The condition $c_n\geqq 0$ is therefore equivalent to

$$\sin (n+1)\theta \sin (n+2)\theta \geqq 0 \qquad (n \geqq 0). \tag{2}$$

Now (2) will fail to hold if and only if there is an integer s such that $(n+1)\theta<s\pi<(n+2)\theta$, that is $n+1<s\pi/\theta<n+2$. In other words (2) fails provided that $s\pi/\theta$ is not an integer for some s. Clearly this will happen unless $\theta=\pi/k$, where k is integral. This completes the proof of the theorem.

Also solved by M. I. Aissen, C. E. Buell, Fritz Herzog, J. B. Kelly, the Proposer, and one whose solution is unsigned.

Contraction Point of a Transformation

4581 [1954, 199]. *Proposed by Ky Fan, University of Notre Dame*

Let M be a metric space in which every closed spheroid, *i.e.*, set of the form

$$S(z_0; r) = \{z\in M \mid d(z_0, z) \leqq r\}, \qquad (z_0 \in M, r > 0)$$

is compact (d denotes the distance). Let f be a continuous transformation from M into itself. Suppose that there is a point $x_0\in M$ such that

$$d(x_0, f(x)) < d(x_0, x) \tag{1}$$

for every $x\neq x_0$ of M. Prove

(i) If $f(x_0)=x_0$, then

$$\lim_{n\to\infty} f^n(x) = x_0 \tag{2}$$

for all $x\in M$; where $f^1(x)=f(x)$ and $f^n(x)=f(f^{n-1}(x))$.

(ii) If $f(x_0)\neq x_0$, then for every point $x\in M$, there is a positive integer k, depending on x, such that $f^k(x)=x_0$.

Solution by R. A. Struble, Illinois Institute of Technology

We show first that for each $x\in M$, if $f^n(x)\neq x_0$ for $n=1, 2, \cdots$, then (2) holds. In fact, the monotonically decreasing sequence $\{d(x_0, f^n(x))\}$ converges to some $R\geqq 0$. Further, since the sequence $\{f^n(x)\}$ is contained in the compact spheroid $S(x_0; d(x_0, x))$, it possesses at least one limit point y. Certainly,

$d(x_0, y)=R$. If $R>0$, then by (1), $d(x_0, f(y))<R$. However, this is impossible since f is continuous at y and every neighborhood of y contains some of the $f^n(x)$ for which $d(x_0, f^n(x))\geqq R$. Hence $R=0$ and (2) is true.

(i) If $f(x_0)=x_0$, then in any event we have (2).

(ii) If $f(x_0)\neq x_0$, then since f is continuous at x_0, (2) cannot be satisfied by any $x\in M$ and hence for each $x\in M$ there is a positive k, depending on x, such that $f^k(x)=x_0$.

Also solved by Joseph Auslander, J. D. Baum, R. R. Bernard, B. F. Bryant, Helen F. Cullen, D. O. Ellis, C. D. Gorman, R. H. Kasriel, P. R. Kelly, A. E. Livingston, M. D. Marcus, H. F. Mattson, O. W. Rechard, A. I. Rosenfeld, W. R. Smythe, Jr., G. H. M. Thomas, L. E. Ward, Jr., G. L. Weiss and R. B. Kellogg, and the Proposer.

Editorial Note. C. D. Gorman calls any point $x_0\in M$ for which (1) holds when $x\neq x_0$, a *contraction point* of M. He proves that there cannot be more than two contraction points of M; he proves also that if $f(x_0)\neq x_0$, M contains at least as many components as there are distinct points in the sequence $\{f^n(x_0)\}$.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

1. *Technical Mathematics.* Harold S. Rice and Raymond M. Knight. McGraw-Hill. 1954. 14+748 pages. $6.50.
2. *Functional Mathematics.* Book 1. W. Gager, C. Carlton, C. Shuster and F. Kokomoor. Charles Scribner's Sons. 14+434 pages.
3. *Functional Mathematics.* Book 2. W. Gager, C. Carlton, C. Shuster and F. Kokomoor. Charles Scribner's Sons. 15+447 pages.

Technical Mathematics, written "especially for junior colleges, technical institutes and other schools where students are being prepared for employment as engineering technicians equipped with the ability to apply fundamental mathematics to the problems of industry," covers arithmetic and methods of computation, intuitive geometry, introductory algebra, trigonometry, analytic geometry, and vector algebra.

The book is divided into seven teaching units. In addition, there are three appendices on Computation Aids and Approximations, Formulas of Geometry and Mensuration, and Solution of Higher Degree Equations. The selection of topics and manner of presentation are traditional, as would be expected in a text of this kind, and it contains materials for two or three semesters. An early introduction to the slide rule and methods of approximations should be of great benefit to any student who will but make use of these aids. An excellent chapter

on tables and interpolation is made somewhat confusing by the absence of the tables (squares and circumferences) referred to. A large number of applied problems is a feature of this text, the problems being of wide selection and appeal. Teachers using this text should have no difficulty in answering the usual question: "But what is the *use* of this?"

The two volumes in the *Functional Mathematics* series are apparently intended for the first two years of high school, although there seems to be no explicit statement to that effect. The reviewer found mention of a *Book 3* in the preface, and feels that the number of volumes, and the grades they are intended for, should appear somewhere on the cover or the title page.

The use of the word "functional" in two different senses is unfortunate. For example, on page iv of the preface is the statement, "Mathematics is the study of functional relationships," and a few sentences later, "It is evident, therefore, that functional mathematics is mathematics which functions in the daily life of the student."

The contents of these two volumes include the materials of the traditional courses for the ninth and tenth grades, with such additional topics as installment buying, life insurance, saving and investing, taxes, and considerable material on computation with approximate numbers. Features of these books are a "maintenance program" of spiral learning, a high degree of applications to life situations, and an easy, informal style, written to the student. Teachers in high school who have broken away from the traditional compartmentalized courses will welcome these books.

M. D. EULENBERG
Wright Junior College
Chicago, Illinois

Infinity. By Lillian R. Lieber. New York, Rinehart & Company, Inc. 1953. 10+359 pages. $5.00.

It was both a genuine pleasure and a profitable experience to read this book, and nothing that may be said farther along in this review is to be taken as unfavorable criticism. Every student of mathematics from the elementary calculus level up should read it, and many of the keener students below that level will find it useful.

Printed in the usual Lieber style of short lines of varying length to "make reading easier," the book first introduces the reader to SAM; the S standing for science, making observations, testing, gathering facts from the outside world, and realism in general; the A representing intuition, hunches, emotions, imagination, and postulations; and the M signifying logical reasoning, mathematics. The author appeals to the reader in a sort of refrain, as the chapters develop, to make full and well-balanced use of the SAM within him, not only in his study of mathematics itself, but also in all of life.

After a series of illustrations given to show that infinity is not "something very large," the author deals with the mathematical point of view, showing how

mathematicians first began with the idea of a "potential" infinity and later developed the notion of "actual" infinity in the form of sets, each having an infinite "number" of members.

Potential infinity is illustrated by means of "division" by zero, the extension of a line without limit, and a study of conic sections in which it is shown that the parabola has just one ideal point and the hyperbola two. Non-Euclidean geometries are clearly and interestingly treated, and the terms hyperbolic, elliptic, and parabolic as applied to geometry are explained.

In the treatment of "actual" infinity, Cantor's theory of sets is discussed at great length and with good effect. Different kinds of sets are defined, described, and illustrated by means of examples. After introducing transfinites, the author goes into a discussion of operations with them. Some of the postulates governing transfinite cardinals and ordinals are explained and illustrated.

There is at about this point somewhat of a digression to acquaint the reader with the concepts of the fourth and higher dimensions. This is done by means of the binomial theorem, cutting a square to show that $(a+b)^2=a^2+2ab+b^2$, a cube to illustrate $(a+b)^3$, and thus raising the question as to how $(a+b)^4$, *etc.*, could be presented geometrically. The bare statement that the binomial theorem was originated by Newton is in the opinion of this reviewer hardly correct.

The calculus of Newton and Leibnitz is nicely done, including the relation between differentiation and integration and an explanation of the fundamental theorem of integral calculus. The convenience of the calculus for finding areas bounded partly or entirely by curves is taken up after an exposition of the difficulties the ancient Greeks had with this type of problem. Archimedes' quadrature of the parabola is used to illustrate.

It would have been appropriate and extremely interesting, especially when explained in the Lieber style, to have included what is usually called "The Method" of Archimedes for discovering new theorems. Archimedes wasn't far from the integral calculus when he applied his method to the segment of a parabola, for his scheme of summing lines to get an area was very similar to our idea of slices in integration.

The author concludes her work with a discussion of modern integration and some of the tough problems in the theory of transfinites. The various types of integrals developed by Riemann, Stieltjes, Lebesgue, and Denjoy are mentioned, with the most attention given to functions that are Riemann integrable.

There are some 25 or 30 pages of art work that adds a great deal to make the book interesting and challenging. To this reviewer, some of the sketches were also quite bewildering—as was probably intended.

It was not the purpose of this review to search for minor errors, but if there had been many they would certainly have been noticed. One slip was observed on page 20 where in a footnote the length of an inch in centimeters was given as 2.4.

F. W. KOKOMOOR
University of Florida

NEW BOOKS RECEIVED

Algebra for College Students. By W. W. Whyburn and P. H. Daus. New York, Prentice-Hall. 1955. xi+290 pages.

Mathematics of Engineering Systems. By D. F. Lawden. Published in America by John Wiley and Sons, New York. 1955. viii+380 pages. $5.75.

Les Fondements Logiques des Mathematiques. By E. W. Beth. Paris, Gauthier-Villars. 1955. xv+241 pages. $7.38.

Axiomatique Intuitionniste Sans Negation de la Geometrie Projective. (No. 6 in Collection de Logique Mathematique, Series A). Paris, Gauthier-Villars. 1954. 108 pages. $3.72.

Existence Theorems for Ordinary Differential Equations. By F. J. Murray and K. S. Miller. New York, New York University Press. 1954. x+154 pages. $5.00.

Traite de Mecanique Rationnelle. By Paul Appell. Paris, Gauthier-Villars. 1955. 202 pages.

Memorial des Sciences Mathematiques, Le Calcul Symbolique a deux variables et ses applications. By L. Poli and P. Delerue. Paris. 1954. 77 pages.

Cours de Geometrie Infinitesimale. By Gaston Julia. Paris, Gauthier Villars. 1955. 80 pages.

Functional Mathematics. Book 3. By W. A. Gager, L. C. Lyle, C. N. Shuster and F. W. Kokomoor. New York, Charles Scribner's Sons. 1955. 13+481 pages. $3.20.

Introduction to Modern Algebra and Matrix Theory. By R. A. Beaumont and R. W. Ball. New York, Rinehart and Co. 1954. xii+331 pages. $6.00.

Contributions to the Theory of Partial Differential Equations. Edited by L. Bers, S. Bochner, and F. John. Princeton, New Jersey, Princeton University Press. 1955. 257 pages. $4.00.

Psychological Statistics, Second Edition. By Quinn McNemar. New York, John Wiley and Sons, Inc. 1955. vii+408 pages. $6.00.

Algèbre. By Paul Dubreil. Paris, Gauthier-Villars. 1954. 467 pages. $11.71.

Trigonometry. By Roy Dubisch. New York, The Ronald Press Company. 1955. 395 pages. $5.00.

Oeuvres de Marie Sklodowska Curie. Edited by Irene Joliot Curie, Warsaw, Academie Polonaise des Sciences, 1954. xii+685 pages.

Contributions to the Solution of Systems of Linear Equations and the Determination of Eigenvalues. U. S. Department of Commerce, National Bureau of Standards Applied Mathematics Series 39. Edited by Olga Taussky. Washington 25, D.C., Government Printing Office. 139 pages. 1954. $2.00.

Exploring Mars. By Robert S. Richardson. New York, McGraw-Hill Book Company, Inc. $4.00.

Engineering Cybernetics. By H. S. Tsien. New York, McGraw-Hill Book Company. 1954. xii+289 pages. $6.50.

La Theorie Harmonique. By Andre Lamouche. *Le Principe de Simplicite dans les Mathematiques et dans les Sciences Physiques.* Paris, Gauthier-Villars. 1954. $5.43.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

ANNUAL MEETING OF THE ASSOCIATION FOR COMPUTING MACHINERY

The annual general meeting of the Association for Computing Machinery will take place at the Moore School of Electrical Engineering, University of Pennsylvania, September 14–16, 1955. As in the past, this meeting is intended to serve both as a place for the reporting of new ideas and developments in the applications of computing machinery and as a place for renewing old friendships and making new ones.

CONFERENCE FOR TEACHERS OF MATHEMATICS AT UNIVERSITY OF CALIFORNIA AT LOS ANGELES

The Fifth Annual Conference for Teachers of Mathematics, together with a Fourth Annual Mathematics Laboratory, will be held at the University of California at Los Angeles, July 5 to 15, 1955. The University Extension will present the sessions in cooperation with the Departments of Mathematics and Education of the University, the California Mathematics Council, and the National Council of Teachers of Mathematics.

The purpose of the Los Angeles conference is to bring together teachers interested in mathematics—arithmetic through calculus—to study problems in the teaching of mathematics and to learn new uses of mathematics in various fields of endeavor. For further information, write to University of California Extension, Los Angeles 24.

FELLOWSHIPS FOR SECONDARY SCHOOL MATHEMATICS TEACHERS

Case Institute of Technology has announced that thirty fellowships for secondary school mathematics teachers are available. These fellowships are designed to pay the expenses of attending a program which will be held at the Institute, June 19 to July 29, 1955. This program is sponsored by the E. I. du Pont de Nemours Company.

Courses of study in the program together with faculty members are as follows: Professor Morris, modern ideas in geometry; Professors McCuskey and Crowder, the calculus and its applications; Professor Brown and assistants, modern methods of computation. Inquiries should be addressed to: Dean Elmer Hutchisson, Case Institute of Technology, 10900 Euclid Avenue, Cleveland 6, Ohio.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1955:

State University of Iowa. June 14 to August 10: Professor Conkwright, differential equations; Professor Dye, introduction to topology; Professor Hogg, elements of statistics; Professor Muhly, survey of elementary college mathematics; Professor Oberg, elementary theoretical mechanics, numerical integration of differential and integral equations; Professor Woods, studies in secondary algebra, constructive geometry; additional courses scheduled are advanced calculus, matrices and determinants, elementary theory of numbers, reading in mathematics, and research.

University of California at Los Angeles. June 20 to July 30: Professor Sokolnikoff, tensor analysis; Staff, seminar in mathematics, seminar in numerical analysis.

University of Delaware. June 20 to July 29: Professor Remage, fundamentals of geometry. (Graduate credit available for teachers.)

University of Michigan. A number of courses in applied statistics and statistical methods will be given under the auspices of the Departments of Mathematics (see this MONTHLY, April, 1955), Economics, Sociology, Psychology, and the School of Business Administration. Also, the following special programs will be offered: July 18 to August 12, Program in Survey Research Techniques; June 20 to August 12, Summer Institute in Mathematics for Social Scientists; August 1 to 12, Program on Digital Computers; August 17 to 27, Program on Quality Control. Inquiries regarding courses in statistics should be addressed to Professor C. C. Craig, Department of Mathematics, and inquiries regarding allied programs should be addressed to the Office of the Summer Session.

The University of Michigan announces also its Third Annual Workshop for College Professors, June 20 to July 8, 1955. There will also be a one-week Institute on College Administration, July 11 to 15. The University will also offer a six-week course entitled, The College Teacher. The Workshop and the Institute are directed by Professor A. D. Henderson.

University of Minnesota. June 13 to July 16: Professor Roy of University of North Carolina, multivariate analysis, experimental designs for research workers; Professor Loud, methods of applied mathematics; Professor Nering, advanced algebraic theory, analytic projective geometry. July 18 to August 20: Professor Hatfield, development of the number system, special functions; Professor Loud, methods of applied mathematics.

University of Pittsburgh. June 13 to July 22 and July 25 to August 2: Professor Laush (first session) and Professor Blumberg (second session), differential equations; Professor Christiano, advanced calculus; Professor Taylor, functions of a complex variable; Professor Laush, functions of a real variable; Professor Bryson, partial differential equations, Fourier series; Professor Levine, topology; Professor Bompiani, non-euclidean geometry. July 5 to August 12: Professor

Knipp, algebra for teachers, solid analytic geometry; Professor Myers, recreational mathematics for teachers; Professor Levine, theory of equations; Mr. Kachun, navigation for teachers. August 15 to August 26: Mr. Sebesta, geometry for teachers. June 20 to August 12 (evenings): Professor Bryson, Laplace transform theory and applications.

College teachers of mathematics may work for advanced degrees at the University of Pittsburgh by taking course work in the summer and doing reading and research during the academic year at their own institutions.

University of Texas. June 7 to August 31: Staff, advanced calculus with engineering applications, differential equations with engineering applications. June 7 to July 18: Professor Ettlinger, advanced calculus, differential equations and applications; Professor Greenwood, interpolation and graphical methods; Professor Guy, Fourier and Laplace transforms; Professor Moore, introduction to the foundations of geometry, theory of sets; Dr. Nicol, introduction to abstract algebra and number theory; Staff, conference course, thesis, dissertation. July 19 to August 31: Professor Cooper, theory of functions of a complex variable; Professor Lane, advanced calculus, introduction to the applications of continued fractions; Professor Lubben, introduction to modern projective geometry, topics in modern algebra; Professor Wall, analytic functions, infinite processes; Staff, differential equations with applications, thesis, dissertation.

University of Virginia. June 20 to August 13: Professor Botts, differential equations, applied mathematics; Professor Ball, transformation theory; Professor McShane, advanced analysis. July 5 to August 13: Professor Botts, foundations of algebra.

Wayne University. The Computation Laboratory announces four special summer courses: June 6 to June 11, electronic computers, business and engineering applications; June 13 to June 18, automatic data processing; June 20 to June 25, mathematical programming of management problems; June 27 to July 2, numerical methods and advanced programming techniques.

PERSONAL ITEMS

Rutgers University reports: Dr. R. K. Brown, formerly in military service, has been appointed to an instructorship; Professor O. H. Alisbah of the University of Ankara, Turkey, has been Visiting Professor for the academic year 1954–1955.

Professor A. A. Albert of the University of Chicago has been reappointed as a member of the General Sciences Panel Advisory to the Assistant Secretary of Defense for Research and Development, D. A. Quarles, for the year 1955.

Mr. H. W. Baker, formerly a student at Nebraska Wesleyan University, is now a teaching assistant in the Department of Chemistry, Purdue University.

Mr. W. R. Brittenham, previously a graduate student at the University of Wisconsin, Milwaukee, has a position as a mathematician at the A. O. Smith Corporation, Milwaukee, Wisconsin.

Mr. R. L. Brooks, formerly a mathematician with the National Bureau of Standards, Washington, D. C., is now employed as a flight test analysis engineer by Lockheed Aircraft Corporation, Burbank, California.

Mr. G. C. Bush, previously a student at McMaster University, has been appointed to a research assistantship at Massachusetts Institute of Technology.

Dr. K. H. Carlson of Michigan State College has been appointed to an assistant professorship at Valparaiso University.

Professor D. R. Carpenter of Roanoke College has retired with the title of Professor Emeritus.

Professor J. C. Cothran of the Department of Chemistry, University of Minnesota, Duluth, has been appointed Lecturer at Kansas State Teachers College, Emporia.

Dr. M. D. Davis of the Institute for Advanced Study has been appointed to an assistant professorship at the University of California, Davis.

Dr. W. A. Horning, formerly a theoretical physicist for the Hanford Works, Richland, Washington, is now a physicist for the Ramo-Wooldridge Corporation, Los Angeles, California.

Mr. R. L. Huckins, previously a teaching fellow at the University of Wichita, has been promoted to an instructorship.

Mr. P. M. Moskowitz, recently a dynamics engineer for Sikorsky Aircraft, Bridgeport, Connecticut, has a position as a programmer for Remington-Rand Corporation, New York City.

Mr. L. R. Norwood, formerly a senior engineer at the Electronic Defense Laboratory, Sylvania Electric Products, Mountain View, California, has a position as an administrative engineer and member of the technical staff, Hughes Research and Development Laboratories, Culver City, California.

Dr. R. H. Owens, previously in the Office of Naval Research, Pasadena, California, has been transferred to Washington, D. C.

Dr. T. D. Riney of Purdue University has accepted a position with the Bell Telephone Laboratories, Allentown, Pennsylvania.

Professor C. E. Rusch of Mission House College has been appointed to an assistant professorship at Wisconsin State College, Eau Claire.

Mr. W. D. Serbyn, formerly a graduate assistant at the Carnegie Institute of Technology, is now a part-time instructor at the Institute of Technology, University of Minnesota.

Dr. R. G. Stoneham of the University of California has been appointed a mathematician with Logistics Research, Inc., Redondo Beach, California.

Professor Irving Sussman of the University of San Francisco is on leave of absence for the year 1954–55 and is an associate professor at California State Polytechnic College.

Mr. C. J. Vanderlin, Jr., formerly a teaching assistant at the University of Wisconsin, Extension Division, has been promoted to an instructorship.

Dr. R. S. Varga, previously a teaching fellow at Harvard University, has a

position as a senior scientist for Westinghouse Atomic Power Division, Pittsburgh, Pennsylvania.

Professor Emeritus G. W. Gorrell of the University of Denver died on January 9, 1954. He was a member of the Association for thirty-three years.

Professor Emeritus J. M. Kinney of Wilson Junior College, Chicago, Illinois, died on January 19, 1955. He was a charter member of the Association.

Professor Emeritus H. F. Price of Pacific University died on January 12, 1955.

Mr. A. C. Washburne, Actuary Emeritus of the Berkshire Life Insurance Company, died on August 26, 1954. He was a charter member of the Association.

Assistant Professor Harold Weintaub of Tufts College died on November 7, 1954.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

REPORT OF THE JOINT COMMITTEE OF THE AMERICAN SOCIETY FOR ENGINEERING EDUCATION AND THE MATHEMATICAL ASSOCIATION OF AMERICA ON ENGINEERING MATHEMATICS

This Committee was appointed in the fall of 1953 and charged with:

1. Consideration of how mathematics can be taught in the most effective way for engineers;
2. Acting as liaison between ASEE and the MAA;
3. Keeping in close contact with the MAA Committee on the undergraduate mathematical program under the Chairmanship of W. L. Duren.

The matter of liaison and of keeping in contact with Duren's committee have been effected in the following ways. The first meeting of the Committee was in Baltimore on December 30, 1953 in conjunction with the annual meeting of the MAA. The second meeting of the Committee took place in Urbana on June 17, 1954 at the annual meeting of the ASEE. The MAA Committee on the Undergraduate Program was represented at our first meeting by W. L. Duren and at the second by G. B. Price. At the invitation of R. S. Burington, Chairman of the Mathematics Division of the ASEE, Price also presented to the Division a report on current efforts to revise the undergraduate program in mathematics.

The liaison between the MAA and the ASEE is probably best maintained through the Mathematics Division of the ASEE. Mathematicians engaged in teaching engineers should be encouraged to join and to support the activities of both the ASEE and the MAA. Continued publicity about the activities of the Mathematics Division of the ASEE in the MONTHLY should be encouraged.

Mathematics for Engineers. The Committee has directed its attention to the following three questions in this connection.

1. What mathematics should be taught?
2. Who should teach it?
3. How should it be taught?

What Mathematics should be taught to engineering students? This question is the subject of study and debate from many sides.

The ASEE's Committee on Evaluation of Engineering Education in its Interim Report of June 15, 1954 (p. 10) states: "At the undergraduate level, competence in the theory and use of simple ordinary differential equations and their application to the solution of physical problems lies close to the boundary of minimum acceptability of mathematics in any satisfactory engineering curriculum. For students whose interests will be centered in research, development, or the higher phases of analysis and design, or who contemplate subsequent graduate study in engineering, additional mathematics may be both desirable and necessary."

The MAA's Committee on the Undergraduate Mathematics program has endeavored to formulate a first year mathematics course which would "try to incorporate into collegiate mathematical teaching the valuable results of modern researches in mathematics and logic and to eliminate that which is unnecessary or faulty." While they have in mind a program for *all* students who study mathematics in college, they are thoroughly aware of the fact that "the revision must concentrate upon the needs of the vast majority ('95 per cent') of students who take mathematics to increase their powers for solving problems and grasping the ideas of science. Thus the approach to mathematics at this stage (*i.e.* during the first two years of college) shall be *operational* rather than merely *appreciative.*" They are experimenting with, and currently engaged in writing text materials for, a basic three-hour course for all students who take mathematics for the first year. In the first semester this would take up: graphs; functions; limits; introductory calculus based on polynomials, exponentials, and logarithms. The material in the second semester would include: sets, mathematical language, groups, combinatorics, distributions, approximate sums, probability. The basic course would be supplemented for engineering and physical science students with a two-hour course as follows:

First semester

Numerical trigonometry, graphical analysis, Newton's methods, numerical integration, tables of logarithms and exponential functions, slide rule, log and semi-log paper, rate problems.

Second semester

Worded problems, binary representation, computing machines, centroids and moments of inertia, averages, precision in measure and computation, finite differences, tables of normal probability density, sampling.

For engineering students, this would be continued in the second year into a course in calculus and analytic geometry with geometric and mechanical applications using vector methods, which would also include linear differential equations with constant coefficients. The program has been tried, with success, at Tulane.

There are other experimental mathematics programs being tried at the University of Washington, Haverford College, Carleton College, The Illinois Institute of Technology, Brown University, Yale, and no doubt at many other institutions.

High School Preparation. So far as the undergraduate curriculum in engineering is concerned, the crucial question seems to be "how much useful mathematics can be taught to the engineers in the time now allotted to mathematics in the undergraduate curriculum?" The answer to this question obviously depends upon the previous mathematical preparation of the entering freshmen. Where this preparation is meager, it is now customary for the students to take college algebra, trigonometry, and analytic geometry in the freshman year and to follow this with a course in calculus in the sophomore year. Where the high school preparation includes trigonometry as well as a strong background in algebra, it is possible to give a combined course in calculus and analytic geometry beginning in the freshman year.

But not all entering freshmen engineering students will have had strong preparation in high school mathematics. There is reason to hope that significant improvements can be made in our methods of taking care of these students. Williams College has been successful in taking students without trigonometry directly into the unified calculus, analytic geometry, and differential equations course and developing the necessary analytic trigonometry as it is needed in the calculus. The universal freshman program envisioned by the MAA Committee on the Undergraduate Program will presuppose only one year of high school algebra and one year of geometry. A program in mathematics and physics was devised at Lafayette College for students enrolled in the preinduction program sponsored by the Ford Foundation. Most of these men entered college after two years of high school. They had studied neither physics nor trigonometry and their mathematical background was primarily limited to one year of algebra and to one year of plane geometry. The two year program at Lafayette included the essential parts of trigonometry, analytic geometry, calculus, and differential equations. Experiments along these lines while not conclusive do at least suggest the possibility of by-passing some of the traditional prerequisites for the calculus with a resulting acceleration and expansion of the useful mathematics that can be included in the normal engineering curriculum. However, some serious questions have been raised in connection with these experimental programs. The major items of concern seem to be:

1. that the existence of the experimental programs, with the low level of mathematical pre-requisites for some of them, may result in a further

weakening of the high school mathematics preparation of future engineers and scientists. It is taken as axiomatic that high school students will not learn as much mathematics in two years as they would in four. In the face of increasing demands for mathematical competence, the colleges and universities (so the argument goes) should, if anything, be upgrading their entrance requirements in mathematics for students in engineering and the physical sciences. And the University of Illinois, for one, seems to be moving in this direction.

2. that the experimental courses may attempt to survey so many topics that they will attempt to teach "a little about everything and not much about anything." This may be an exaggerated fear, but it nevertheless exists.
3. that "by far and away the largest portion of engineers get along on and *use* a fair amount of the elements of algebra, plane, solid, and analytic geometry, trigonometry in particular, plus the fundamental concepts of the calculus and copious use of handbooks and tables. Moreover, whatever the shortcomings of the traditional first year mathematics courses in engineering may be, they have at least made a serious attempt to meet the student where his knowledge left off and usually provide time for review and filling in gaps."

This Committee has not had the opportunity to debate the issues raised here. Accordingly, we make no recommendation either for or against any of the experimental programs now under discussion. We do, however, endorse experimentation as such. Moreover, we feel that the prospect for curriculum improvement is enhanced by a lively and serious debate of these issues.

We also wish to call attention to an experiment of a somewhat different nature, designed to stimulate gifted students to make more progress in mathematics at an early age. The Ford Foundation through its Fund for the Advancement of Education has recently sponsored a study called *The School and College Study of Admission with Advanced Standing*. The Mathematics Committee of that study proposed a program in mathematics for the more able students which would include a strong introduction to differential and integral calculus in the *twelfth grade*. And, in a similar vein, we wish to call attention to a recent talk by R. S. Burington on "Mathematics for our time," reported in the Mathematics Teacher, vol. XLVII, No. 5, pp. 295–298, May 1954.

What conclusions, then, has this Committee reached in answer to the question "What mathematics should be taught to engineers"? We wrote personal letters to about forty mathematicians, scientists, and engineers soliciting their comments and advice on the question of what mathematics should be taught to engineers, and how it could be taught most effectively. From the replies received we deduced that:

a. There was a consensus of opinion that all engineers would benefit by studying mathematics through at least the elementary course in ordinary differential equations.

On the question of specific course content we are not prepared to take a stand either for or against the program proposed by the MAA Committee; largely because it is still in the experimental stage and because of certain doubts stated above.

b. Statistics and probability deserve more attention than they now receive. It is not at all obvious how this is to be managed within the present time limitations. One possible solution to the problem has been suggested by the MAA Committee on undergraduate curriculum. This problem deserves further study.

c. It would be highly desirable to provide the opportunity for students to take further electives in mathematics; for example advanced calculus, complex variables, numerical analysis, matrix algebra, and so on.

Who should teach mathematics to engineers? There is, of course, the possibility of a dichotomy with some people answering "the engineering staff, in order to increase motivation" and with others answering "the mathematics staff because of their deeper insight into the logical structure of the subject." In his talk at Urbana, L. W. Cohen referred to mathematics as a language. In order to learn to use this language effectively engineers need teachers who know the language thoroughly and know how to use it. In this connection we quote from the interim report of the Committee on Evaluation of Engineering Education of June 15, 1954: "A minimum level of performance in mathematics should be established whether it be obtained in required mathematics or in engineering courses. However, few engineering courses are taught in a manner to make a significant contribution to the students' knowledge of basic mathematics, nor is time available for this purpose. The engineering sciences and subsequent professional subject matter should be developed by making effective use of such mathematical proficiency, and should be taught by staff members competent to do so."

We are in complete agreement that, in these days when some engineers need far more mathematics than the minimum now required, it is important that mathematics be taught by mathematicians who by their inspiration and enthusiasm can stimulate an interest in mathematics *per se*. Unfortunately, however, the impression exists that some engineering students are taught by mathematicians who have little or no appreciation for the interests and needs of the engineers. We believe that those mathematicians who are responsible for curriculum planning should consult with their engineering colleagues from time to time to try to determine the engineers' current and anticipated needs in mathematics. Then they should try to meet those needs by providing appropriate courses taught by competent, sympathetic teachers.

One experiment which has been tried, and which appears to be worth wider consideration, is for the mathematics department to invite a member of the engineering staff to teach one or more regular mathematics courses. This gives him a chance to become familiar with the structure and timing of the mathe-

matics course, and also puts him in an excellent position to make criticisms and offer suggestions on the course from the point of view of an engineer. This might also be done on a reciprocal basis with a member of the mathematics staff teaching one of the engineering sciences in order to become familiar with the applications of mathematics in it. One member of our Committee has reported that he took part in such an experiment at the University of Illinois and found the experience valuable, particularly the discussions among these involved in the exchange teaching. This sort of an exchange arrangement would be one way of helping a teacher to avoid getting into a rut. Other ways include: individual or group research projects, consulting on mathematics problems arising in industry or governmental laboratories, sabbatical leaves, and so on.

There are two aspects of motivation that bear further mention. On the one hand, mathematics should continually draw upon the physical and engineering sciences for illustrative problems. But, of equal or perhaps greater importance is the fact that motivation is tremendously enhanced if liberal use is made of mathematical techniques in physics and the engineering sciences, both in the first two years and in the later years. We quote again from the interim report previously cited (p. 11): "In the engineering sciences, full use should be made of the prerequisite mathematics, physics, and chemistry, recognizing that repetition is a normal pedagogical necessity, but that it can be most effective only when consciously and purposefully used. Perhaps nowhere else can the qualities of a scholarly engineering faculty be employed so effectively as in the presentation of these engineering sciences with an appropriate mathematical understanding."

How shall mathematics be taught to engineering students? These students are interested in mathematics as a tool. The majority of them are not content to study mathematics for its own sake. They usually can be induced to become interested in "why" a technique works, but only after they are convinced that: (a) it is a useful technique, and (b) they understand how to apply it. Teachers recognize this fact by using a large amount of problem material. In this connection, we should like to call attention to the collection of *Engineering Problems Illustrating Mathematics*, prepared by a committee of the ASEE under the chairmanship of John W. Cell, and published by the McGraw-Hill Book Company in 1943. These problems are divided into categories illustrating the mathematics usually covered in college algebra, trigonometry, analytic geometry, and differential and integral calculus. It has been suggested that more material of this kind would be helpful. In particular, more examples where the emphasis is on the analysis of the problem and the methods for setting it up in mathematical form, would be desirable. A fund of such problems requiring statistical techniques would be particularly welcomed by some teachers. Our deliberations and the replies to our inquiries have indicated needs for:

a. More emphasis on numerical methods.
b. More emphasis on graphical methods.

c. More emphasis on fundamental concepts.
d. More emphasis on translation from physical problems to mathematical problems, and on the interpretation of the results of the solution of the mathematical problems in terms of the given physical problems.
e. More use of mathematics in junior and senior engineering courses.

Much has been published on the question of improvement of teaching methods. The Committee feels it is appropriate to list here, for convenient reference, the following books and articles which are related to the work of the Committee:

BOOKS AND ARTICLES RELEVANT TO THE WORK OF THE COMMITTEE

J. W. Cell, *Engineering Problems Illustrating Mathematics*, McGraw-Hill (1943).
Effective Teaching—McGraw-Hill (1950) (Fred C. Morris, Editor)
Report of the Sub-Committee on Minimum Essentials in Mathematics for Engineering Instruction-California Committee for the Study of Education.
Report of the Committee on Adequacy and Standards of Engineering Education, J. of Engineering Education, vol. 42, No. 5 (1952) pp. 249–254.
Proceedings of the A.S.E.E.

Vol. 39, 1931–32	pp. 299–310
42, 1934–35	149–152, 292–296
45, 1937–38	122–131, 190–194, 548–558
46, 1938–39	716–724
47, 1939–40	394–401, 699–703
49, 1941–42	57– 66, 346–352
50, 1942–43	432–437
51, 1943–44	664–668
52, 1944–45	407–413
54, 1946–47	330–335, 531–535, 536–539, 641–652
55, 1947–48	175–180, 300–307, 308–312, 358–365, 366–373
58, 1950–51	308–310
59, 1951–52	170–172
60, 1952–53	33– 46, 136–144, 472–475

Conclusion. One imagines that the question of "how to teach (anything)" most effectively confronts every conscientious teacher almost constantly. One also feels that the question of *motivation* on the part of the teacher is at least as important as it is for his students. If he feels that *what* he is teaching is interesting, important, esthetic, and useful—then his enthusiasm will almost certainly stimulate his students to *learn*. But no teacher can long maintain a fictitious enthusiasm for subject matter which, to him, is dull, trivial, non-esthetic, and largely useless. It is for this reason that the committee chose to include a consideration of *what* mathematics to teach as well as *how* to teach it. We do not propose any definite answers to either question. But we have pointed out some

experiments which seem to us to be significant. In doing so, we hope not only to recognize and encourage those who are conducting these experiments, but also to stimulate others to devise their own or to try some of the existing ones.

Recomendations. We recommend the following to the ASEE and MAA and their respective members involved in the teaching of mathematics to engineers:

1. Use all appropriate means to encourage better mathematics preparation in the high schools.
2. Teach "operational" rather than merely "appreciative" mathematics to engineers in the first two years of college.
3. Understand the engineers' needs and try to meet them; whatever the level of preparation.
4. Debate vigorously the issues involved in curriculum structure.
5. Make a serious attempt to introduce statistics and probability into the curriculum in the first two years.

Respectfully submitted:
R. S. BURINGTON
J. W. CELL
R. P. DILWORTH
W. E. RESTEMEYER
S. E. WARSCHAWSKI
G. B. THOMAS, JR., Chairman

FORD FOUNDATION GRANT

The Ford Foundation has awarded a grant of $25,000 to the Mathematical Association of America for a study by the Association's Committee on the Undergraduate Program in Mathematics. This study will review the status of teaching and research utilization in mathematics, and the relationships of mathematics to the sciences and other fields of knowledge.

The grant will be used by the Committee to facilitate communication among those who are working on these problems. The Committee plans to bring together writing teams which will seek to bridge the gap between modern mathematics and the undergraduate curriculum by writing pilot text materials. However the Committee does not plan to use its funds to subsidize the writing of actual text books.

The present chairman of the Committee on the Undergraduate Program is Professor E. J. McShane of University of Virginia. Other members of the Committee are: J. G. Kemeny, Dartmouth College; G. B. Price, University of Kansas; A. L. Putnam, University of Chicago; A. W. Tucker, Princeton University; W. L. Duren Jr., Tulane University, *ex officio*.

THE JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The seventeenth annual meeting of the Northern California Section of the Mathematical Association of America was held at the University of California, Berkeley, on January 15, 1955. Professor J. G. Herriot, Chairman of the Section, presided at both the morning and the afternoon sessions.

One hundred three persons attended the meeting including the following fifty-eight members of the Association:

H. L. Alder, H. A. Arnold, H. M. Bacon, R. A. Barnett, G. C. Barton, Alice K. Bell, Stoughton Bell, II, M. T. Bird, R. L. Blair, D. W. Blakeslee, W. E. Bleick, S. G. Bourne, Randolph Church, T. H. Dewey, Roy Dubisch, D. G. Duncan, Hazel E. Eggett, G. C. Evans, E. J. Farrell, F. D. Faulkner, Ruth A. Fish, Harley Flanders, L. C. Graue, Hazel M. Hadley, W. R. Hanson, J. G. Herriot, Marjorie L. Hoffman, V. E. Hoggatt, Jr., H. D. Huskey, Walter Jennings, J. L. Kelley, R. M. Lakness, C. M. Larsen, D. H. Lehmer, A. R. Lovaglia, R. B. Merkel, A. B. Mewborn, E. D. Miller, F. R. Morris, W. H. Myers, Andrewa R. Noble, C. D. Olds, C. L. Perry, Jr., J. P. Pierce, M. H. Protter, F. M. Pulliam, C. H. Rawlins, Jr., R. M. Robinson, E. B. Roessler, J. P. Roth, Abraham Seidenberg, Mary V. Sunseri, Irving Sussman, J. V. Talacko, C. C. Torrance, H. G. Tucker, L. A. Walker, A. R. Williams.

At the business meeting the following officers were elected for the coming year: Chairman, Professor C. C. Torrance, U. S. Naval Postgraduate School, Monterey; Vice-Chairman, Professor H. L. Alder, University of California, Davis; Secretary-Treasurer, Professor C. D. Olds, San Jose State College.

By invitation of the Section, Professor David Gilbarg, Indiana University and Stanford University, delivered an address at the morning session entitled *Free Surface Flows*. Abstract of this address follows:

The hydrodynamical problem of flows with free boundaries is surveyed from the point of view of the mathematical methods used in solving both special flow problems and problems of general theory. In particular, the method of conformal mapping is described and applied to the determination of the Helmholtz flow past a flat plate, and the comparison method initiated by Lavrentieff is applied to the proof of uniqueness of symmetric Helmholtz flows past curved obstacles.

The following papers were presented:

1. *On mean values and harmonic polynomials*, by Professor C. L. Perry, Jr., U. S. Naval Postgraduate School, Monterey.

The Gauss mean value property (average over circles or spheres) was discussed (Kellogg, *Potential Theory*, p. 224). The results of Beckenbach and Reade (*Trans. A.M.S.*, vol. 53, and *Duke Math. Jour.*, vol. 12), Perkins (*Proc. Int. Cong. of Math.*, 1954), and Walsh (*Bull. A.M.S.*, vol. 42), who investigated mean values over regular polygons and polyhedra, were reviewed. Properties of functions having similar mean value relations for irregular configurations were also considered.

2. *A classroom note on the carpenter's square*, by Professor M. T. Bird, San Jose State College.

The locus described by the interior angle of the carpenter's square as the outer edges of the arms remain in contact with two fixed points $B\ (-a, 0)$ and $A\ (a, 0)$ is sought. Parametric, rectangular coordinate, and polar equations may not identify the locus as a limaçon. A mechanical model produces a curve which has an axis of symmetry oblique to the coordinate axes. This inductive approach suggests a coordinate system with pole at the double point and polar axis on the axis of symmetry. The polar equation of the limaçon is established by elementary geometry of the circle.

3. *A remark on Hilbert's nullstellensatz*, by Professor Abraham Seidenberg, University of California, Berkeley.

Let (1): $F_1(X_1, \cdots, X_n)=0, \cdots, F_s(X_1, \cdots, X_n)=0, G(X_1, \cdots, X_n)\neq 0$ be a system of polynomial equations and inequalities over a field K. By regarding all the X_i but one as parameters,

one sees how to eliminate that one from (1), hence how to eliminate all the X_i, and thus decide whether (1) has a solution. If (1) has no solution, the decision method shows how to find an integer r and polynomials $A_1, \cdots, A_s$ such that $G^r = A_1F_1 + \cdots + A_sF_s$. This proof of Hilbert's theorem is so trivial that it can be applied with little modification to algebraic partial differential systems.

4. *Dyadic treatment of rotation*, by Professor F. M. Pulliam, U. S. Naval Postgraduate School, Monterey.

In a combined treatment of vector algebra and solid analytic geometry, the rotation of axes can be more completely covered than in the standard texts on analytic geometry. A simple introduction to dyadics is shown to be sufficient to get the axis of rotation and the angle of rotation from the direction angles of the new cartesian axes and conversely. The dyadic method is natural and easy and might at least be given in the appendix of such a combined treatment.

5. *Analytic geometry from the vector point of view*, by Professor S. G. Bourne, University of California, Berkeley.

From a mathematical point of view, the vector approach to coordinate geometry is superior to the conventional approach, in that concepts do not depend on the dimensionality of the space, and proofs are simplified. From a practical point of view, its superiority rests in the early introduction of vectors into the student's mathematical education, thereby enabling him to utilize the algebra of vectors in his study of the physical sciences and engineering. This treatment is suitable either for a first course or a second course in analytic geometry. The needs of the modern physical sciences are such that an early introduction of these mathematical methods is deemed indispensable.

6. *Fourier kinematic analysis of the reciprocating engine by contour integration*, by Professor W. E. Bleick, U. S. Naval Postgraduate School, Monterey.

The exact Fourier kinematic analyses of the motion of the piston and connecting rod of a reciprocating engine were obtained by contour integration. The simplicity of this exact method compares favorably with that of the usual approximate truncated power series approach. The proper selection of branch cuts is involved. This may be a useful classroom example illustrating the application of contour integration to a simple problem in engineering mechanics.

7. *On the reduction of a matrix to diagonal form*, by Professors M. P. Epstein and Harley Flanders, University of California, Berkeley, presented by Professor Flanders.

This paper has been published in this MONTHLY, vol. 62, pp. 168–171.

8. *The classification of problems in the calculus of variations*, by Professor F. D. Faulkner, U. S. Naval Postgraduate School, Monterey.

The problems usually studied in a beginning course in calculus of variations belong to a class known as the problem of Bolza [G. A. Bliss, *Lectures on the Calculus of Variations*, Chicago, 1946, §68, 69]. If these are expressed as a problem of Mayer, and the differential equations written in the particular form $du_i = K_i(x, y, u_j)$, $i, j = 1, 2, \cdots n$, where x is a preferred variable in the sense that it is monotonic, it becomes clear what type of end conditions can be imposed: they cannot, in general, involve y. The *order* of the system is defined as the order of the resulting differential equation for y and is generally equal to n, but may be less. It determines the number of parameters in the family of extremals through a point.

C. D. OLDS, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-sixth Summer Meeting, University of Michigan, Ann Arbor, Michigan, August 29–30, 1955.

Thirty-ninth Annual Meeting, Rice Institute, Houston, Texas, December 30, 1955.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May 7, 1955.
ILLINOIS, Monmouth College, Monmouth, May 13–14, 1955.
INDIANA, Butler University, Indianapolis, May 7, 1955.
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, College of St. Teresa, Winona, Minnesota, May 7, 1955.
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA
OHIO
OKLAHOMA
PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 17, 1955.
PHILADELPHIA
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, University of Buffalo, May 14, 1955.
WISCONSIN, Cardinal Stritch College, Milwaukee, May 7, 1955.

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These Monographs are a series of expository books intended to make topics in pure and applied mathematics accessible to teachers and students of mathematics and also to non-specialists and scientific workers in other fields. One copy of each Monograph may be purchased by members of the Association for $1.75 each. Additional copies and copies for non-members of Monographs 1–8 are priced at $3.00 each, and must be purchased from the Open Court Publishing Co., LaSalle, Illinois. In the case of Monographs 9 and 10, additional copies and copies for non-members must be purchased at $3.00 from John Wiley and Sons, 440 Fourth Ave., New York 16, N. Y. The more recent numbers are:

No. 3. *Mathematical Statistics* by H. L. Rietz, ix+181 pages.
No. 4. *Projective Geometry* by J. W. Young, ix +185 pages.
No. 5. *History of Mathematics in America before 1900* by D. E. Smith and Jekuthiel Ginsburg, viii+210 pages.
No. 6. *Fourier Series and Orthogonal Polynomials* by Dunham Jackson, xiv+234 pages.
No. 7. *Vectors and Matrices* by C. C. MacDuffee, xi+192 pages.
No. 8. *Rings and Ideals* by N. H. McCoy, xii +216 pages.
No. 9. *The Theory of Algebraic Numbers* by Harry Pollard, xii+143 pages.
No. 10. *The Arithmetic Theory of Quadratic Forms* by B. W. Jones, x+212 pages.

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By GNEDENKO AND KOLMOGOROV; translated by K. L. CHUNG, *Syracuse University*, with additional appendix material provided by the translator and J. L. DOOB, *University of Illinois*

A systematic and definitive account of the most important part of classical probability theory. Much of the material has hitherto been available only in periodicals and in Russian. It is intended as a standard reference work on probability theory which may be used as a text or supplementary reading for advanced courses on the subject. Parts of the book may also serve as a text for an introductory course in probability, or as a text for courses in statistics.

The exposition combines great generality and simplicity. Beyond a rigorous knowledge of calculus, the book is formally self-contained. There are two new appendixes in this translation; one on measure theory by J. L. Doob, the other on unimodal distributions by the translator.

*Cloth, 264 pp., 1954 — $8.50 ($7.50)**

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By WILFRED KAPLAN, *University of Michigan*

FROM A REVIEW IN *The Scientific Monthly:*
"This book offers all one could wish for in a text in advanced calculus. It is clearly written, rigorous, and comprehensive. The material is selected so as to make the text serve well for a course designed for engineers as well as for students of pure mathematics. The author presents vector analysis early in the book and uses the vector methods to give a clear treatment of the differential and integral calculus in space . . . Much is added to the appearance of the book by its beautiful illustrations . . . Here is indeed an excellent advanced calculus book."

Cloth, 679 pp., 241 illus., 1st ed. 2nd ptg., 1953 — $9.50 ($8.50)

INTRODUCTION TO MEASURE AND INTEGRATION

By M. E. MUNROE, *University of Illinois*

FROM A REVIEW IN *Mathematical Reviews:*
"This is a textbook designed to 'present measure theory from the abstract or postulational point of view and yet to do this in such a way that the graduate student as well as the expert will find the work helpful.' The treatment is modern without being forbiddingly 'highbrow,' and the author has been at great pains to smooth the reader's path, both by frequent informal explanations and by judicious concatenation of theorems and lemmas. The result is an easily readable textbook that is admirably adapted to its purpose. The book is lavishly provided with examples, which range from simple illustrative problems to theorems (with hints of proofs) that extend or broaden the theory given in the text."

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By SAM PERLIS, *Purdue University*

FROM A REVIEW IN *American Mathematical Monthly:*
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Contents: Introduction. The Arc Length Function. The Sine and Cosine Functions. Tables and Graphs. Inverse Sine and Cosine Functions. Other Trigonometric Functions. Identities and Equations. Trigonometric Functions of Angles. Identities and Equations for Functions of Angles. Solution of Triangles. Complex Numbers and Trigonometry. Definition and Properties of Logarithms. Applications of Logarithms. *Appendixes. Special Protractor included. 119 ills., tables. 396 pp.*

ANALYTIC GEOMETRY

Alfred L. Nelson, Karl W. Folley, William M. Borgman
—all of Wayne University

This textbook, for use in a freshman course in analytic geometry, is planned as a preparation for the calculus rather than as a study of geometry. To aid future students of the calculus, basic sciences, and engineering, emphasis is placed on the problems of finding the equation from a geometric description of a locus, and describing a curve from a given equation of a locus. *114 ills., tables. 215 pp.*

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R. B. Kershner, *The Johns Hopkins University; and*
L. R. Wilcox, *Illinois Institute of Technology*

This book fills a long existing need for a treatise on the axiomatic method. It also serves as a reference source for workers in those sciences which are increasingly employing the results and techniques of abstract mathematics. It introduces the reader to ideas and methods that pervade mathematical research—aiming to bridge the gap between classical and modern methods. *Illus. 416 pp.*

THE NATURE OF NUMBER

—An Approach to Basic Ideas of Modern Mathematics

Roy Dubisch, *Fresno State College*

Understandable, over-all view of what modern mathematics is about; the nature of its theory, wherein ideas largely replace the more familiar arithmetical concepts; and the types of abstract problems that present-day mathematicians are working on. In engaging manner, the book leads the reader from basic arithmetic into the realm of higher mathematical thought. *24 ills. 159 pp.*

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